

# Spaces of Initial Data for Differential Equations in Hilbert Spaces and the Kato problem

R.V. Shamin \*

## 1 Introduction

Consider the Cauchy problem for the abstract parabolic problem

$$u'(t) + \mathcal{A}u(t) = 0, \quad t > 0, \quad (1.1)$$

$$u(0) = \varphi \in H, \quad (1.2)$$

where  $\mathcal{A} : H \rightarrow H$  is a generator of an analytic semigroup in a Hilbert space  $H$ .

In this paper we deal with the important question of the existence of so-called strong solutions for problem (1.1), (1.2) (see the definition in Section 2). It is well-known [1, 2] that if the operator  $\mathcal{A}$  is not bounded, then there exist initial functions  $\varphi \in H$  such that the problem (1.1), (1.2) has no strong solutions. The set of all  $\varphi \in H$  such that the problem has a strong solution is called a space of initial data for the problem (1.1), (1.2).

A study of the space of initial data for the problem (1.1), (1.2) is closely related to the well-known Kato problem. In [3] Kato formulated the following question: do the domains of the operators  $\mathcal{A}^{1/2}$  and  $(\mathcal{A}^*)^{1/2}$  coincide? In [4] it is shown that, in general, the answer is negative. However, if  $\mathcal{A}$  is an elliptic operator with measurable coefficients, then the answer is positive (see [5] and references therein).

In the present work we introduce a wide class of coercive operators  $\mathcal{A}$  such that the domains of  $\mathcal{A}^{1/2}$  and  $(\mathcal{A}^*)^{1/2}$  coincide. This class includes in particular differential-difference operators, functional differential operators with contracted and expanded arguments, and others. (Notice that some functional differential operators were also studied by T. Kato and J.B. McLeod [6].)

The paper is organized as follows. In Sections 2 and 3 we consider the statement of the problem and define the space of initial data. The main result

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on the interpolation of the domains of coercive operators is obtained in Section 4. In Section 5 we apply the abstract result of Section 4 to parabolic functional differential equations and consider some examples.

Notice that earlier parabolic functional differential equations were studied in [7, 8]. In [9] it was shown that the domains of differential-difference operators might contain non-smooth functions, which makes a study of initial data spaces for the corresponding parabolic problems quite difficult.

Under some additional restrictions on the domain  $\mathcal{D}(\mathcal{A})$  of an elliptic differential-difference operator and on the domain  $Q \subset \mathbb{R}^n$  (on which the initial functions are defined), the space of initial data for the corresponding parabolic differential-difference equation was investigated in [10]. Our new approach allows to describe the space of initial data for various classes of parabolic functional differential equations without any additional restrictions on  $\mathcal{D}(\mathcal{A})$  and the domain  $Q$ .

A brief summary of the main results was given in [11].

## 2 Statement of problem

Let  $V$  and  $H$  be separable Hilbert spaces. Suppose  $V$  is dense in  $H$  and the embedding  $V \subset H$  is continuous. Denote by  $V'$  the adjoint space. Clearly, we have  $V \subset H \subset V'$ .

We consider a bounded linear operator  $A : V \rightarrow V'$ .

**Definition 2.1** *The operator  $A$  is called  $V$ -coercive if for each  $v \in V$*

$$\operatorname{Re}\langle Av, v \rangle \geq c_1 \|v\|_V^2, \quad (2.1)$$

where  $c_1 > 0$  does not depend on  $v$ .

**Remark 2.1** *By Theorem 9.1 [12, Chapter 1] the operator  $A$  has a bounded inverse operator.*

We consider the semilinear form  $a(u, v)$  in  $H$  given by  $a(u, v) = \langle Au, v \rangle$ ,  $\mathcal{D}(a) = V$ . By virtue of (2.1), the form  $a$  is a closed sectorial form in  $H$ . Indeed, let  $u \in V$ ; then we have

$$|\operatorname{Im} a(u, u)| \leq |a(u, u)| \leq c_2 \|u\|_V^2.$$

From this and (2.1) it follows that the numerical range  $\Theta(a)$  satisfies

$$\Theta(a) \subset \{\lambda \in \mathbb{C} : -\theta_1 < \arg \lambda < \theta_1\},$$

where  $\theta_1 = \arctg(c_2/c_1) < \pi/2$ .

We introduce the operator  $\mathcal{A} : H \rightarrow H$ , generated by the form  $a$ , (i.e.,  $(\mathcal{A}u, v)_H = a(u, v)$ ,  $u \in \mathcal{D}(\mathcal{A})$ ,  $v \in V$ ). Obviously,  $\mathcal{A}u = Au$  whenever  $u \in \mathcal{D}(\mathcal{A})$ . By virtue of Theorem 2.1 [13, Chapter 6], the unbounded operator  $\mathcal{A} : H \rightarrow H$  is a closed, dense, sectorial one.

We consider the Hilbert space  $\mathcal{D}(\mathcal{A})$  with the inner product  $(u, v)_{\mathcal{D}(\mathcal{A})} = (\mathcal{A}u, \mathcal{A}v)_H + (u, v)_H$ .

We introduce the Hilbert space  $\mathcal{W}(\mathcal{A}) = \{w \in L_2(0, T; \mathcal{D}(\mathcal{A})) : w' \in L_2(0, T; H)\}$  with the inner product

$$(u, v)_{\mathcal{W}(\mathcal{A})} = \int_0^T (u', v')_H dt + \int_0^T (\mathcal{A}u, \mathcal{A}v)_H dt + \int_0^T (u, v)_H dt.$$

Here we consider the derivatives in the sense of distributions in  $L_2(0, T; H)$ .

Consider the differential equation in the Hilbert space  $H$

$$u'(t) + \mathcal{A}u(t) = f(t) \quad (t \in (0, T)) \quad (2.2)$$

with the initial condition

$$u|_{t=0} = \varphi, \quad (2.3)$$

where  $0 < T < \infty$ ,  $f \in L_2(0, T; H)$ , and  $\varphi \in H$ .

**Definition 2.2** A function  $u \in \mathcal{W}(\mathcal{A})$  satisfying (2.2), (2.3) is called a strong solution of problem (2.2), (2.3).

### 3 Strong solvability

**Lemma 3.1** Let  $\mathcal{A}$  be a coercive operator. Then the operator  $-\mathcal{A}$  is a generator of an analytic semigroup  $\{T_t\}_{t \geq 0}$  in  $H$ .

*Proof:* Since the operator  $\mathcal{A}$  is a closed, dense, and sectorial one, by virtue of Theorem 1.24 [13, Chapter 9], the operator  $-\mathcal{A}$  is a generator of an analytic semigroup  $T_t$  in  $H$ . ■

We introduce the set  $\Phi = \{\varphi \in H : \text{problem (2.2), (2.3) has a strong solution}\}$ .

**Definition 3.1** The set  $\Phi$  is called a space of initial data for problem (2.2), (2.3).

We have  $\mathcal{D}(\mathcal{A}) \subset \Phi \subset H$ . The space of initial data for problem (2.2), (2.3) can be described by means of an interpolation space. So let us introduce the conception of interpolation spaces.

Let  $H_1$  be a Hilbert space, dense in  $H$ , with the continuous embedding  $H_1 \subset H$ . Let  $t[u, v]$  be a closed, symmetric, and positive form in  $H$ , defined by the formula  $t[u, v] = (u, v)_{H_1}$ ,  $\mathcal{D}(t) = H_1$ . By Theorem 2.23 [13, Chapter 6], there exists a positive operator  $T$  such that  $\mathcal{D}(T^{1/2}) = \mathcal{D}(t) = H_1$ . Put  $\Lambda = T^{1/2}$ . We introduce the interpolation spaces  $[H_1; H]_\theta$  ( $0 \leq \theta \leq 1$ ) by formula  $[H_1; H]_\theta = \mathcal{D}(\Lambda^{1-\theta})$  ( $0 \leq \theta \leq 1$ ). The space  $[H_1; H]_\theta$  is a Hilbert space with the inner product

$$(u, v)_{[H_1; H]_\theta} = (\Lambda^{1-\theta}u, \Lambda^{1-\theta}v)_H + (u, v)_H.$$

**Theorem 3.1** *Suppose the operator  $A$  is  $V$ -coercive.*

*Then problem (2.2), (2.3) has a unique strong solution if and only if  $\varphi \in [\mathcal{D}(\mathcal{A}), H]_{1/2}$ . Moreover, this solution is given by*

$$u(t) = T_t\varphi + \int_0^t T_{t-s}f(s)ds, \quad (3.1)$$

where  $\{T_t\}_{t \geq 0}$  is the analytic semigroup generated by the operator  $-\mathcal{A}$ .

*Proof:* By virtue of Theorem 3.7 [2, Chapter 1], problem (2.2), (2.3) has a unique strong solution if and only if the following inequality holds:

$$\int_0^T \|\mathcal{A}T_t\varphi\|_H^2 dt < \infty. \quad (3.2)$$

At the same time, Theorem 1.14.5 [14, Chapter 1] implies that inequality (3.2) takes place if and only if  $\varphi \in [\mathcal{D}(\mathcal{A}), H]_{1/2}$ . ■

Theorem 3.1 shows that  $\Phi = [\mathcal{D}(\mathcal{A}), H]_{1/2}$ . Clearly, one would like to get some constructive description of the space  $[\mathcal{D}(\mathcal{A}), H]_{1/2}$ . Generally, this is a very difficult problem since the domain of  $\mathcal{A}$  may have quite a complicated form.

In the next Section we obtain a natural (and rather general) condition on the domain  $\mathcal{D}(\mathcal{A})$  of the operator  $\mathcal{A}$ , under which we have  $\Phi = V$ .

## 4 Interpolation of domain of coercive operator

We consider the adjoint operator  $A^* : V \rightarrow V'$ . We define the unbounded operator  $\mathcal{A}' : \mathcal{D}(\mathcal{A}') \subset H \rightarrow H$  with the domain  $\mathcal{D}(\mathcal{A}') = \{u \in V : A^*u \in H\}$  by the formula  $\mathcal{A}'u = A^*u$ . If the operator  $A$  is  $V$ -coercive, then the operator  $A^*$  is  $V$ -coercive, too. For all  $u \in \mathcal{D}(\mathcal{A})$ ,  $v \in \mathcal{D}(\mathcal{A}')$  we have

$$(\mathcal{A}u, v)_H = \langle Au, v \rangle = \langle u, A^*v \rangle = (u, \mathcal{A}'v)_H. \quad (4.1)$$

Hence  $\mathcal{A}' \subset \mathcal{A}^*$  and  $\mathcal{A} \subset (\mathcal{A}')^*$ . Since the operators  $A$  and  $A^*$  are  $V$ -coercive, we have  $0 \notin \sigma(\mathcal{A}) \cup \sigma(\mathcal{A}')^1$ . By virtue of Lemma 13, [15, §6, Chapter 14] concerning adjoint operators, we have  $\mathcal{A}^* = \mathcal{A}'$ .

**Theorem 4.1** *Let the operator  $A$  be  $V$ -coercive. If the embeddings  $V \subset [\mathcal{D}(\mathcal{A}); H]_{1/2}$  and  $V \subset [\mathcal{D}(\mathcal{A}'); H]_{1/2}$  are continuous, then  $V = [\mathcal{D}(\mathcal{A}); H]_{1/2} = [\mathcal{D}(\mathcal{A}^*); H]_{1/2}$ .*

*Proof:* For any  $u \in H$  the form  $(\mathcal{A}w, u)_H$  defines the linear continuous functional  $\langle w, f \rangle = (\mathcal{A}w, u)_H$  on  $\mathcal{D}(\mathcal{A})$  since

$$\sup_{w \in \mathcal{D}(\mathcal{A})} \frac{|(\mathcal{A}w, u)_H|}{\|w\|_{\mathcal{D}(\mathcal{A})}} \leq \sup_{w \in \mathcal{D}(\mathcal{A})} \frac{\|\mathcal{A}w\|_H \|u\|_H}{(\|\mathcal{A}w\|_H^2 + \|w\|_H^2)^{1/2}} \leq \|u\|_H.$$

<sup>1</sup> $\sigma(\mathcal{A})$  (resp.  $\sigma(\mathcal{A}')$ ) stands for the spectrum of the operator  $\mathcal{A}$  (resp.  $\mathcal{A}'$ ).

The functional  $f$  has the form  $A'_0 u = f$  with  $A'_0 : H \rightarrow (\mathcal{D}(\mathcal{A}))'$ . From the inequality  $\|f\|_{(\mathcal{D}(\mathcal{A}))'} \leq \|u\|_H$ , it follows that the operator  $A'_0$  is bounded.

Now we shall demonstrate that  $\mathcal{A}^* \subset A'_0$ . Suppose that  $u \in \mathcal{D}(\mathcal{A}^*)$  and  $w \in \mathcal{D}(\mathcal{A})$ . Denote  $A'_0 u = f_1$  and  $\mathcal{A}^* u = f_2$ . From (4.1) it follows that

$$\langle w, f_1 \rangle = (\mathcal{A}w, u)_H = (w, \mathcal{A}^* u)_H = \langle w, f_2 \rangle.$$

Since  $w \in \mathcal{D}(\mathcal{A})$  is arbitrary and  $\mathcal{D}(\mathcal{A})$  is dense in  $H$ , we have  $\mathcal{A}^* u = A'_0 u$ . From the inclusion  $\mathcal{A}^* u \in H$  and the embedding  $H \subset (\mathcal{D}(\mathcal{A}))'$ , it follows that  $\mathcal{A}^* u = A'_0 u \in H$ . Since the operator  $\mathcal{A}^* : \mathcal{D}(\mathcal{A}^*) \rightarrow H$  is bounded, the operator  $A'_0 : \mathcal{D}(\mathcal{A}^*) \rightarrow H$  is bounded as well.

By virtue of the interpolation theorem (see Theorem 5.1 [12, Chapter 1]) the operator  $A'_0$  being defined as

$$A'_0 : [\mathcal{D}(\mathcal{A}^*); H]_{1/2} \rightarrow [H; (\mathcal{D}(\mathcal{A}))']_{1/2}$$

is bounded. By virtue of Theorem 6.2 [12, Chapter 1], we have  $[H; (\mathcal{D}(\mathcal{A}))']_{1/2} = ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$ . Therefore the operator

$$A'_0 : [\mathcal{D}(\mathcal{A}^*); H]_{1/2} \rightarrow ([\mathcal{D}(\mathcal{A}); H]_{1/2})' \quad (4.2)$$

is bounded. Let us demonstrate that  $A'_0 u = \mathcal{A}^* u$  for  $u \in V$ . Suppose  $u \in V$  and  $w \in \mathcal{D}(\mathcal{A})$ . Denote  $A'_0 u = f_1$  and  $\mathcal{A}^* u = f_2$ . We have

$$\langle w, f_1 \rangle = (\mathcal{A}w, u)_H = \langle \mathcal{A}w, u \rangle = \langle w, \mathcal{A}^* u \rangle = \langle w, f_2 \rangle.$$

Since  $w \in \mathcal{D}(\mathcal{A})$  is arbitrary, we see that  $\mathcal{A}^* u = A'_0 u$ . From the inclusion  $\mathcal{A}^* u \in V'$  and the embedding  $V' \subset (\mathcal{D}(\mathcal{A}))'$ , it follows that  $\mathcal{A}^* u = A'_0 u \in V'$ .

Take arbitrary  $f \in V'$ . Then we have  $u = (\mathcal{A}^*)^{-1} f \in V$  and

$$\|u\|_V \leq c_2 \|f\|_{V'}, \quad (4.3)$$

where  $c_2 > 0$  does not depend on  $f$ . By the assumption we have  $u \in [\mathcal{D}(\mathcal{A}^*); H]_{1/2}$  and

$$\|u\|_{[\mathcal{D}(\mathcal{A}^*); H]_{1/2}} \leq c_3 \|u\|_V, \quad (4.4)$$

where  $c_3 > 0$  does not depend on  $f$ .

From (4.2) it follows that  $A'_0 u = f \in ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$ . Inequalities (4.3) and (4.4) imply

$$\|f\|_{([\mathcal{D}(\mathcal{A}); H]_{1/2})'} \leq c_4 \|u\|_{[\mathcal{D}(\mathcal{A}^*); H]_{1/2}} \leq c_4 c_3 \|u\|_V \leq c_4 c_3 c_2 \|f\|_{V'}.$$

Therefore we have continuous embedding  $V' \subset ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$ . By the assumption the space  $([\mathcal{D}(\mathcal{A}); H]_{1/2})'$  is continuously embedded in  $V'$ . Thus we have  $V' = ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$  and, therefore,  $V = [\mathcal{D}(\mathcal{A}); H]_{1/2}$ .

Similarly, one obtains  $V = [\mathcal{D}(\mathcal{A}^*); H]_{1/2}$ . ■

Since the operator  $-\mathcal{A}$  is a generator of an analytic semigroup, one can define fractional powers of the operator  $\mathcal{A}$  (see [3], [16]).

**Theorem 4.2** *Suppose all the conditions of Theorem 4.1 hold.*

*Then  $\mathcal{D}(\mathcal{A}^{1/2}) = \mathcal{D}((\mathcal{A}^*)^{1/2}) = V$ .*

*Proof:* We take arbitrary number  $\alpha \in (0, 1/2)$ . By virtue of Theorem 1.15.2 [14, Chapter 1], the operator  $\mathcal{A}^\alpha : [\mathcal{D}(\mathcal{A}), H]_{1/2} \rightarrow [\mathcal{D}(\mathcal{A}), H]_{1/2-\alpha}$  is bounded and invertible. From Theorem 4.1 it follows that the operator

$$\mathcal{A}^\alpha : V \rightarrow [\mathcal{D}(\mathcal{A}), H]_{1/2-\alpha} \quad (4.5)$$

is bounded.

For any  $\alpha \in (0, 1/2)$  we introduce the operator  $K_\alpha = \Lambda^{1/2-\alpha} \mathcal{A}^\alpha$ , where  $\Lambda$  is a positive operator with the domain  $\mathcal{D}(\Lambda) = \mathcal{D}(\mathcal{A})$ . From (4.5) it follows that the operators  $K_\alpha : V \rightarrow H$  are bounded for all  $\alpha \in (0, 1/2)$ .

For any  $\varphi \in \mathcal{D}(\mathcal{A})$  we have

$$\lim_{\alpha \rightarrow 1/2} K_\alpha \varphi = \lim_{\alpha \rightarrow 1/2} \Lambda^{1/2-\alpha} \mathcal{A}^\alpha \varphi = \lim_{\alpha \rightarrow 1/2} \Lambda^{1/2} \Lambda^{-\alpha} \mathcal{A}^{\alpha-1/2} \mathcal{A}^{1/2} \varphi = \mathcal{A}^{1/2} \varphi. \quad (4.6)$$

Here we used Theorem 14.1 [16, Chapter 4] concerning the analyticity of the family of the operators  $\mathcal{A}^{-t}$ ,  $t \geq 0$ . Since  $\mathcal{D}(\mathcal{A})$  is dense in  $H$ , from (4.6) and the Banach-Steinhaus theorem it follows that the operator

$$\lim_{\alpha \rightarrow 1/2} K_\alpha = \mathcal{A}^{1/2} : V \rightarrow H$$

is bounded. Therefore  $V \subset \mathcal{D}(\mathcal{A}^{1/2})$ . Let us show that  $\mathcal{D}(\mathcal{A}^{1/2}) \subset V$ . From (4.5) it follows that the operators  $K_\alpha^{-1} = \mathcal{A}^{-\alpha} \Lambda^{-(1/2-\alpha)} : H \rightarrow V$  are bounded. For any  $\varphi \in \mathcal{D}(\mathcal{A})$  we have

$$\lim_{\alpha \rightarrow 1/2} K_\alpha^{-1} \varphi = \lim_{\alpha \rightarrow 1/2} \mathcal{A}^{-\alpha} \Lambda^{-(1/2-\alpha)} \varphi = \mathcal{A}^{-1/2} \varphi.$$

Hence the operator  $\mathcal{A}^{-1/2} : H \rightarrow V$  is bounded, and we have  $\mathcal{D}(\mathcal{A}^{1/2}) \subset V$ . So we proved that  $\mathcal{D}(\mathcal{A}^{1/2}) = V$ . Since the operator  $\mathcal{A}^*$  has the same properties as  $\mathcal{A}$ , analogously to the above we derive that  $\mathcal{D}((\mathcal{A}^*)^{1/2}) = V$ . ■

Thus Theorem 4.2 is a positive answer for the Kato problem in our case.

## 5 Spaces of initial data

Let  $\mathcal{A}$  be the operator defined in Section 2. Let  $C : V \rightarrow H$  be a bounded operator. We introduce the unbounded operator  $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset H \rightarrow H$  acting by the formula  $\mathcal{L}u = \mathcal{A}u + Cu$ ,  $u \in \mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A})$ .

We consider the parabolic equation

$$u'(t) + \mathcal{L}u(t) = f(t) \quad (t \in (0, T)) \quad (5.1)$$

with the initial condition

$$u(0) = \varphi. \quad (5.2)$$

**Definition 5.1** A function  $u \in \mathcal{W}(\mathcal{L})$  satisfying (5.1), (5.2) is called a strong solution for problem (5.1), (5.2).

**Theorem 5.1** Let the operator  $\mathcal{A}$  be such that the conditions of Theorem 4.1 hold.

Then the operator  $-\mathcal{L}$  is a generator of an analytic semigroup  $\{T_t\}_{t \geq 0}$ . Moreover, for any  $f \in L_2(0, T; H)$  problem (5.1), (5.2) has a unique strong solution if and only if  $\varphi \in V$ . This solution is given by

$$u(t) = T_t \varphi + \int_0^t T_{t-s} f(s) ds.$$

*Proof:* At first let us prove that the operator  $-\mathcal{L}$  is a generator of an analytic semigroup. From Lemma 3.1 it follows that the operator  $-\mathcal{A}$  is a generator of an analytic semigroup. Since  $C : V \rightarrow H$  is bounded and  $\mathcal{A}$  has a bounded inverse, we have

$$\|Cu\|_H \leq c_1 \|u\|_V \leq c_2 \|\mathcal{A}u\|_H$$

for all  $u \in \mathcal{D}(\mathcal{A})$ . Therefore, by virtue of the theorem on perturbation of generators of analytic semigroups, the operator  $-\mathcal{L} = -\mathcal{A} - C$  is a generator of an analytic semigroup  $\{T_t\}_{t \geq 0}$ ,  $\|T_t\| \leq M$ , where  $M$  is some positive constant.

Similarly to the proof of Theorem 3.1 we obtain that problem (5.1), (5.2) has a unique strong solution if and only if  $\varphi \in [\mathcal{D}(\mathcal{L}); H]_{1/2}$ . Since  $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A})$ , by virtue of Theorem 4.1 we have  $[\mathcal{D}(\mathcal{L}); H]_{1/2} = V$ . ■

Further we consider some concrete realizations of the operator  $\mathcal{A}$  (namely, functional differential operators) satisfying the conditions of Theorem 5.1.

To begin with, let us prove one auxiliary result. Let  $H_1, H_2$  be Hilbert spaces. Let  $H_i$  be dense in  $H$  and the embeddings  $H_i \subset H$  continuous ( $i = 1, 2$ ).

**Lemma 5.1** Suppose that the embedding  $H_2 \subset H_1$  is continuous.

Then the embedding  $[H_2, H]_{1/2} \subset [H_1, H]_{1/2}$  is also continuous.

*Proof:* For every  $\psi \in H$  and for all  $t > 0$  we define the functional

$$K(t, \psi; H_1, H) = \inf_{\substack{\psi_0 + \psi_1 = \psi, \\ \psi_0 \in H_1, \\ \psi_1 \in H}} (\|\psi_0\|_{H_1} + t\|\psi_1\|_H).$$

By virtue of Theorem 15.1 [12, Chapter 1], we have  $[H_1, H]_{1/2} = \{\psi \in H : \int_0^\infty t^{-2} K^2(t, \psi; H_1, H) dt < \infty\}$ . Since  $H_2 \subset H_1$ , we have

$$K(t, \varphi; H_1, H) \leq K(t, \varphi; H_2, H).$$

Hence if  $t^{-1}K(t, \varphi; H_2, H) \in L_2(0, \infty)$ , then  $t^{-1}K(t, \varphi; H_1, H) \in L_2(0, \infty)$ . Therefore  $[H_2, H]_{1/2} \subset [H_1, H]_{1/2}$ . ■

Let  $Q \subset \mathbb{R}^n$  be a bounded domain with a Lipschitz boundary  $\partial Q$ . Put  $Q_T = Q \times (0, T)$ .

We denote by  $W_2^k(Q)$  the Sobolev space of complex-valued functions with the norm

$$\|u\|_{W_2^k(Q)} = \left\{ \sum_{|\alpha| \leq k} \int_Q |\mathcal{D}^\alpha u(x)|^2 dx \right\}^{1/2},$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ , and  $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_n^{\alpha_n}$ ,  $\mathcal{D}_i = \frac{\partial}{\partial x_i}$ . Denote by  $\mathring{W}_2^k(Q)$  the closure of the linear manifold  $\dot{C}^\infty(Q)$  (of compactly supported functions, infinitely differentiable in  $Q$ ) in the space  $W_2^k(Q)$ . Put  $W_2^{-k}(Q) = (\mathring{W}_2^k(Q))'$ .

Consider the following case:  $H = L_2(Q)$ ,  $V = \mathring{W}_2^1(Q)$ , and  $V' = W_2^{-1}(Q)$ . Introduce the bounded operator  $A_B : \mathring{W}_2^1(Q) \rightarrow W_2^{-1}(Q)$ , defined by the formula  $A_B u = -\operatorname{div}(B \operatorname{grad} u)$  with a bounded operator  $B : L_2^n(Q) \rightarrow L_2^n(Q)$ , where  $L_2^n(Q) = \prod_{k=1}^n L_2(Q)$ .

Denote  $\mathring{W}_2^{1,n}(Q) = \prod_{k=1}^n \mathring{W}_2^1(Q)$ ,  $W_2^{1,n}(Q) = \prod_{k=1}^n W_2^1(Q)$ .

Assume that the following conditions for the operator  $B$  hold.

**Condition 5.1**  $Bu \in W_2^{-1}(Q)$  whenever  $u \in \mathring{W}_2^1(Q)$ , and the operator  $B : \mathring{W}_2^{1,n}(Q) \rightarrow W_2^{1,n}(Q)$  is bounded.

**Condition 5.2** The operator  $A_B$  is  $\mathring{W}_2^1(Q)$ -coercive.

We introduce the unbounded operator  $\mathcal{A}_B : \mathcal{D}(\mathcal{A}_B) \subset L_2(Q) \rightarrow L_2(Q)$  given by  $\mathcal{A}_B u = A_B u$ ,  $u \in \mathcal{D}(\mathcal{A}_B) = \{u \in \mathring{W}_2^1(Q) : A_B u \in L_2(Q)\}$ .

Consider the parabolic function differential equation

$$u_t(x, t) + \mathcal{A}_B u(x, t) + C u(x, t) = f(x, t) \quad ((x, t) \in Q_T) \quad (5.3)$$

with the boundary condition

$$u|_{\partial Q \times (0, T)} = 0 \quad (5.4)$$

and the initial condition

$$u|_{t=0} = \varphi(x) \quad (x \in Q). \quad (5.5)$$

Problem (5.3)–(5.5) is a particular case of problem (5.1), (5.2).

At first we remark that from Condition 5.2 for the operator  $B$  it follows that Condition 5.2 for the operator  $B^*$  holds. Now let us demonstrate that  $(\mathcal{A}_B)^* = \mathcal{A}_{B^*}$ . Indeed, for all  $u, v \in \dot{C}^\infty(Q)$  we have

$$(\mathcal{A}_B u, v)_{L_2(Q)} = -(B \nabla u, \nabla v)_{L_2^n(Q)} = (u, \mathcal{A}_{B^*} v)_{L_2(Q)}. \quad (5.6)$$

Since  $\dot{C}^\infty(Q)$  is dense in  $\mathring{W}_2^1(Q)$ , and  $\mathcal{A}_B$  is closed (which follows from its  $\mathring{W}_2^1(Q)$ -coerciveness) we see that equality (5.6) holds for all  $u \in \mathcal{D}(\mathcal{A}_B)$ ,  $v \in$

$\mathcal{D}(\mathcal{A}_{B^*})$ . Therefore  $\mathcal{A}_{B^*} \subset (\mathcal{A}_B)^*$  and  $\mathcal{A}_B \subset (\mathcal{A}_{B^*})^*$ . Since the operators  $\mathcal{A}_B$  and  $\mathcal{A}_{B^*}$  are  $\dot{W}_2^1(Q)$ -coercive, it follows that  $0 \notin \sigma(\mathcal{A}_B) \cup \sigma(\mathcal{A}_{B^*})$ . By virtue of Lemma 13 [15, Section 6, Chapter 14] concerning adjoint operators, we have  $(\mathcal{A}_B)^* = \mathcal{A}_{B^*}$ .

Since the operator  $B : \dot{W}_2^{1,n}(Q) \rightarrow W_2^{1,n}(Q)$  is bounded, then the embeddings  $\dot{W}_2^2(Q) \subset \mathcal{D}(\mathcal{A}_B)$  and  $\dot{W}_2^2(Q) \subset \mathcal{D}(\mathcal{A}_{B^*})$  are continuous.

From Theorem 11.6 [12, Chapter 1], we have  $[\dot{W}_2^2(Q), L_2(Q)]_{1/2} = \dot{W}_2^1(Q)$ .

**Remark 5.1** *Theorem 11.6 [12, Chapter 1] was proved in [12] for the domain  $Q$  with smooth boundary. Nevertheless in our case this theorem remains true. The proof is based on Lemma 11.3 [12, Chapter 1]. We should only use the Calderon method of extension of functions from Lipschitz domains instead of the Hestenes method, see Theorem 5 [17, Chapter 6, § 3].*

By virtue of Lemma 5.1 we have  $\dot{W}_2^1(Q) \subset [\mathcal{D}(\mathcal{A}_B); L_2(Q)]_{1/2}$ ,  $\dot{W}_2^1(Q) \subset [\mathcal{D}(\mathcal{A}_{B^*}); L_2(Q)]_{1/2}$ , and the embeddings are continuous. Thus conditions of Theorem 5.1 hold for problem (5.3)–(5.5). Therefore we have the following result.

**Lemma 5.2** *Suppose that the operator  $B$  satisfies Conditions 5.1 and 5.2. Suppose that the operator  $B^*$  satisfies Condition 5.1.*

*Then  $[\mathcal{D}(\mathcal{A}_B); L_2(Q)]_{1/2} = \dot{W}_2^1(Q)$ .*

Now we shall show that Conditions 5.1 and 5.2 are quite natural in theory of functional differential equations.

Let us impose some restrictions on the domain  $Q$ . Suppose  $Q \subset \mathbb{R}^n$  is a bounded domain with boundary  $\partial Q = \bigcup_i \overline{M_i}$  ( $i = 1, \dots, N_0$ ), where  $M_i$  are  $(n-1)$ -dimensional manifolds of class  $C^\infty$  that are open and connected in the topology of  $\partial Q$ . Assume that in a neighborhood of any point  $g \in \partial Q \setminus \bigcup_i M_i$  the domain  $Q$  is diffeomorphic to an  $n$ -dimensional dihedral angle for  $n \geq 3$  and to a plane angle for  $n = 2$ .

We introduce the difference operator  $R : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$  by the formula

$$Ru(x) = \sum_{h \in M} a_h(x)u(x+h).$$

Here  $a_h \in C^\infty(\mathbb{R}^n)$ , the set  $M$  consists of a finite number of vectors  $h \in \mathbb{R}^n$  with integer coordinates.

We introduce the operator  $R_Q = P_Q R I_Q : L_2(Q) \rightarrow L_2(Q)$ , where  $I_Q : L_2(Q) \rightarrow L_2(\mathbb{R}^n)$  is the operator of extension of functions from  $L_2(Q)$  by zero in  $\mathbb{R}^n \setminus Q$ ;  $P_Q : L_2(\mathbb{R}^n) \rightarrow L_2(Q)$  is the operator of restriction of functions from  $\mathbb{R}^n$  to  $Q$ .

Let the operator  $B$  be equal to the operator  $R : L_2^n(Q) \rightarrow L_2^n(Q)$  defined by the formula

$$(Ru)_i = \sum_{j=1}^n R_{ijQ}(u)_j, \quad i = 1, 2, \dots, n.$$

We introduce the unbounded differential-difference operator  $\mathcal{A}_R : \mathcal{D}(\mathcal{A}_R) \subset L_2(Q) \rightarrow L_2(Q)$  acting in the space of distributions  $D'(Q)$  by the formula  $\mathcal{A}_R u = A_R u$ ,  $u \in \mathcal{D}(\mathcal{A}_R) = \{u \in \dot{W}_2^1(Q) : \mathcal{A}_R u \in L_2(Q)\}$ .

To formulate conditions for  $\dot{W}_2^1(Q)$ -coerciveness of the operator  $\mathcal{A}_R$ , we introduce an auxiliary notation, see [9].

Denote by  $G$  the additive group generated by  $M$ . Let  $Q_r$  be the open connected components of the set  $Q \setminus \left( \bigcup_{h \in G} (\partial Q + h) \right)$ .

**Definition 5.2** *The sets  $Q_r$  are called subdomains. The set  $\mathcal{R}$  of all the subdomains  $Q_r$  ( $r = 1, 2, \dots$ ) is called a decomposition of the domain  $Q$ .*

The decomposition  $\mathcal{R}$  can be divided into disjoint classes in the following way:  $Q_{r_1}, Q_{r_2} \in \mathcal{R}$  belong to the same class if there exists an  $h \in G$  such that  $Q_{r_2} = Q_{r_1} + h$ . We denote the subdomains  $Q_r$  by  $Q_{sl}$ , where  $s$  is a number of class and  $l$  is a number of a subdomain in the  $s$ th class. Evidently, each class consists of a finite number  $N = N(s)$  of subdomains  $Q_{sl}$ , and  $N(s) \leq ([\text{diam}Q] + 1)^n$ . A set of classes can be countable.

We define the matrices  $R_s = R_s(x)$  ( $x \in \overline{Q}_{s1}$ ) of orders  $N(s) \times N(s)$  with the elements

$$r_{ij}^s(x) = \begin{cases} a_h(x + h_{si}) & (h = h_{sj} - h_{si} \in M), \\ 0 & (h_{sj} - h_{si} \notin M), \end{cases}$$

where  $h_{si}$  is such that  $Q_{si} = Q_{s1} + h_{si}$ .

Let  $x \in \overline{Q}_{s1}$  be an arbitrary point. Consider all the points  $x^l \in \overline{Q}$  such that  $x^l - x \in M$ . Since the domain  $Q$  is bounded, the set  $\{x^l\}$  consists of a finite number  $I = I(s, x)$  of points ( $I \geq N(s)$ ). We shall enumerate the points  $x^l$  so that  $x^l = x + h_{sl}$  for  $l = 1, \dots, N = N(s)$ ,  $x^1 = x$ , where  $h_{sl}$  satisfies the condition  $Q_{sl} = Q_{s1} + h_{sl}$ . We introduce the  $I \times I$  matrices  $A_{ijs}(x)$  with elements  $a_{lk}^{ijs}(x)$  given by

$$a_{lk}^{ijs}(x) = \begin{cases} a_{ijh}(x^l), & (h = x^k - x^l \in M), \\ 0, & (x^k - x^l \notin M). \end{cases}$$

We define the  $N \times N$  matrix  $R_{ijs}(x)$  so that if  $N < I$ , then  $R_{ijs}(x)$  is obtained from  $A_{ijs}(x)$  by deleting the last  $I - N$  rows and columns. If  $N = I$ , then  $R_{ijs}(x)$  is equal to  $A_{ijs}(x)$ .

On the one hand, by virtue of Theorem 9.2 [9, Section 9], if for all  $s = 1, 2, \dots$ ,  $x \in \overline{Q}_{s1}$ , and  $0 \neq \xi \in \mathbb{R}^n$  the matrices

$$\sum_{i,j=1}^n (A_{ijs}(x) + A_{ijs}^*(x)) \xi_i \xi_j$$

are positive definite, then the operator  $A_R$  (which coincides with  $A_B$ , where  $B = R$ ) is  $\dot{W}_2^1(Q)$ -coercive. On the other hand, if the operator  $A_R$  is  $\dot{W}_2^1(Q)$ -coercive, then by Theorem 9.1 [9, Section 9], the matrices

$$\sum_{i,j=1}^n (R_{ijs}(x) + R_{ijs}^*(x)) \xi_i \xi_j$$

are positive definite for all  $s = 1, 2, \dots$ ,  $x \in \overline{Q}_{s1}$ , and  $0 \neq \xi \in \mathbb{R}^n$ .

We consider the parabolic differential-difference equation

$$u_t(x, t) - \mathcal{A}_R u(x, t) = f(x, t) \quad ((x, t) \in Q_T), \quad (5.7)$$

with the boundary condition

$$u|_{\partial Q \times (0, T)} = 0 \quad (5.8)$$

and the initial condition

$$u|_{t=0} = \varphi(x) \quad (x \in Q). \quad (5.9)$$

**Theorem 5.2** *Suppose that the operator  $\mathcal{A}_R$  is  $\dot{W}_2^1(Q)$ -coercive.*

*Then problem (5.7)–(5.9) has a unique strong solution if and only if  $\varphi \in \dot{W}_2^1(Q)$ .*

*Proof:* The operator  $R$  satisfies Conditions 5.1 and 5.2. Indeed, by virtue of Lemma 8.13 [9, Section 2], the operators  $R, R^*$  maps continuously  $\dot{W}_2^{1,n}(Q)$  into  $W_2^{1,n}(Q)$ . Under assumption the operator  $\mathcal{A}_R$  is  $\dot{W}_2^1(Q)$ -coercive. Therefore, by virtue of Lemma 5.2 and Theorem 3.1, problem (5.7)–(5.9) has a unique strong solution if and only if  $\varphi \in \dot{W}_2^1(Q)$  ■

### Example 5.1

Let us consider problem (5.7)–(5.9), where  $Q = (0, \frac{4}{3}) \times (0, \frac{4}{3})$ ,  $\mathcal{A}_R = -\Delta R_Q$ ,  $R_Q = P_Q R I_Q$ ,  $Ru(x) = u(x) + au(x_1 + 1, x_2 + 1) + bu(x_1 - 1, x_2 - 1)$ ,  $0 < |a + b| < 2$ . Obviously, the decomposition  $\mathcal{R}$  of the domain  $Q$  consists of the two classes of subdomains: 1)  $Q_{11} = (0, \frac{1}{3}) \times (0, \frac{1}{3})$ ,  $Q_{12} = (1, \frac{4}{3}) \times (1, \frac{4}{3})$  and 2)  $Q_{21} = Q \setminus (\overline{Q}_{11} \cup \overline{Q}_{12})$ . We introduce the set  $\mathcal{K} \subset \partial Q$ . The set  $\mathcal{K}$  consists of 4 points:  $g^1 = (\frac{1}{3}, 0)$ ,  $g^2 = (\frac{4}{3}, 1)$ ,  $g^3 = (0, \frac{1}{3})$ ,  $g^4 = (1, \frac{4}{3})$ .

The matrices  $A_s(x)$  ( $x \in \overline{Q}_{s1}$ ;  $s = 1, 2$ ) assume the form

$$A_1(x) = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \quad (x \in \overline{Q}_{11}),$$

$$A_2(x) = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \quad (x \in \overline{Q}_{21} \cap \mathcal{K}), \quad A_2(x) = (1) \quad (x \in \overline{Q}_{21} \setminus \mathcal{K}).$$

Hence the matrices  $(A_s(x) + A_s^*(x))(\xi_1^2 + \xi_2^2)$  ( $x \in \overline{Q}_{s1}$ ;  $s = 1, 2$ ) are positive definite. Therefore the operator  $\mathcal{A}_R$  is  $\dot{W}_2^1(Q)$ -coercive.

By virtue of Theorem 5.2, problem (3.3)–(3.5) has a unique strong solution if and only if  $\varphi \in \dot{W}_2^1(Q)$ .

Unlike elliptic differential equations, smoothness of solutions to the equation

$$\mathcal{A}_R u = f \quad (5.10)$$

can be violated inside the domain  $Q$  even for  $f \in C^\infty(\overline{Q})$ , see [9, Sections 11 and 12]. This means that  $\mathcal{D}(\mathcal{A}_R) \neq \dot{W}_2^1(Q) \cap W_2^2(Q)$ . However, by virtue of

Theorem 11.2 [9, Section 11], if  $u \in \mathcal{D}(\mathcal{A}_R)$  is a solution of equation (5.10) for  $f \in L_2(Q)$ , then  $u \in W_2^2(Q_{sl} \setminus \mathcal{K}^\varepsilon)$  for every  $\varepsilon > 0$  ( $s = 1, 2, \dots; l = 1, \dots, N(s)$ ), where  $\mathcal{K}^\varepsilon = \{x \in \mathbb{R}^n : \rho(x, \mathcal{K}) < \varepsilon\}$ .

In this example the domain  $\mathcal{D}(\mathcal{A}_R)$  cannot be described in the Sobolev space terms. Nevertheless we have got the explicit description of the space of initial data:  $[\mathcal{D}(\mathcal{A}_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$ .

Now we introduce the operators of contraction and expansion of argument  $T_{ij} : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$  and  $T_{ijQ} : L_2(Q) \rightarrow L_2(Q)$  by the formulas

$$T_{ij}u(x) = \sum_{l \in N} a_{ijl}u(q^{-l}x), \quad T_{ijQ}v = P_Q T_{ij} I_Q v,$$

where  $N$  is a finite set of integer numbers;  $a_{ijl} \in \mathbb{C}$ ;  $q > 1$ .

We define the operator  $T : L_2^n(Q) \rightarrow L_2^n(Q)$  by the formula

$$(Tu)_i = \sum_{j=1}^n T_{ijQ} u_j \quad (i = 1, 2, \dots, n),$$

where  $u = (u_1, \dots, u_n)^T$ ,  $Tu = ((Tu)_1, \dots, (Tu)_n)^T$ .

Suppose now that  $B = T$ . We consider the functional differential operator  $A_T$  with contracted and expanded arguments (which coincides with  $A_B$ , where  $B = T$ ). Denote  $t_{ij}(\lambda) = \sum_{l \in N} a_{ijl} \lambda^l$ . By virtue of Theorem 1 [18], if we have

$$\sum_{i,j=1}^n t_{ij}(\lambda) \xi_i \xi_j > 0 \quad (\lambda \in \mathbb{C}, |\lambda| = q^{n/2}, 0 \neq \xi \in \mathbb{R}^n), \quad (5.11)$$

then the operator  $A_T$  is  $\mathring{W}_2^1(Q)$ -coercive. If in, addition, we suppose that

$$\overline{Q} \subset qQ,$$

then, by Theorem 2 [18], condition (5.11) is necessary for  $\mathring{W}_2^1(Q)$ -coercivness of the operator  $A_T$ .

We consider the parabolic functional differential equation with contracted and expanded arguments

$$u_t(x, t) - \mathcal{A}_T u(x, t) = f(x, t) \quad ((x, t) \in Q_T) \quad (5.12)$$

with the boundary condition

$$u|_{\partial Q \times (0, T)} = 0 \quad (5.13)$$

and the initial condition

$$u|_{t=0} = \varphi(x) \quad (x \in Q). \quad (5.14)$$

**Theorem 5.3** *Suppose that condition (5.11) holds.*

*Then problem (5.12)–(5.14) has a unique strong solution if and only if  $\varphi \in \dot{W}_2^1(Q)$ .*

*Proof:* We claim that the operators  $T, T^*$  satisfy Condition 5.1. Indeed, the operators  $T, T^*$  continuously map  $\dot{W}_2^{1,n}(Q)$  into  $W_2^{1,n}(Q)$ . Under the assumptions the operators  $A_T, A_{T^*}$  are  $\dot{W}_2^1(Q)$ -coercive.

Hence, by virtue of Lemma 5.2 and Theorem 3.1, problem (5.12)–(5.14) has a unique strong solution if and only if  $\varphi \in \dot{W}_2^1(Q)$ . ■

**Example 5.2**

Let  $Q = \{|x| < 1\} \subset \mathbb{R}^n$ . We consider the parabolic functional differential equation

$$u_t(x, t) - \Delta(u(x, t) + a_1 u(q^{-1}x, t)) = f(x, t) \quad ((x, t) \in Q_T) \quad (5.15)$$

with the boundary condition

$$u|_{\partial Q \times (0, T)} = 0 \quad (5.16)$$

and the initial condition

$$u|_{t=0} = \varphi(x) \quad (x \in Q) \quad (5.17)$$

where  $q > 1, a_1 \in \mathbb{R}$ .

In this example condition (5.11) means that  $|a_1| \leq q^{1-n/2}$ . By virtue of Theorem 5.3, problem (5.15)–(5.17) has a unique strong solution if and only if  $\varphi \in \dot{W}_2^1(Q)$ .

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