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Spaces of Initial Data for Differential Equations in Hilbert Spaces and the Kato problem

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1 Introduction

Consider the Cauchy problem for the abstract parabolic problem

$$u'(t) + \mathcal{A}u(t) = 0, \quad t > 0, \tag{1.1}$$

$$u(0) = \varphi \in H, \tag{1.2}$$

where $\mathcal{A} : H \to H$ is a generator of an analytic semigroup in a Hilbert space H.

In this paper we deal with the important question of the existence of so-called strong solutions for problem (1.1), (1.2) (see the definition in Section 2). It is well-known [1, 2] that if the operator $\mathcal{A}$ is not bounded, then there exist initial functions $\varphi \in H$ such that the problem (1.1), (1.2) has no strong solutions. The set of all $\varphi \in H$ such that the problem has a strong solution is called a space of initial data for the problem (1.1), (1.2).

A study of the space of initial data for the problem (1.1), (1.2) is closely related to the well-known Kato problem. In [3] Kato formulated the following question: do the domains of the operators $\mathcal{A}^{1/2}$ and $(\mathcal{A}^*)^{1/2}$ coincide? In [4] it is shown that, in general, the answer is negative. However, if $\mathcal{A}$ is an elliptic operator with measurable coefficients, then the answer is positive (see [5] and references therein).

In the present work we introduce a wide class of coercive operators $\mathcal{A}$ such that the domains of $\mathcal{A}^{1/2}$ and $(\mathcal{A}^*)^{1/2}$ coincide. This class includes in particular differential-difference operators, functional differential operators with contracted and expanded arguments, and others. (Notice that some functional differential operators were also studied by T. Kato and J.B. McLeod [6].)

The paper is organized as follows. In Sections 2 and 3 we consider the statement of the problem and define the space of initial data. The main result

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on the interpolation of the domains of coercive operators is obtained in Section 4. In Section 5 we apply the abstract result of Section 4 to parabolic functional differential equations and consider some examples.

Notice that earlier parabolic functional differential equations were studied in [7, 8]. In [9] it was shown that the domains of differential-difference operators might contain non-smooth functions, which makes a study of initial data spaces for the corresponding parabolic problems quite difficult.

Under some additional restrictions on the domain $\mathcal{D}(\mathcal{A})$ of an elliptic differential-difference operator and on the domain $Q \subset \mathbb{R}^n$ (on which the initial functions are defined), the space of initial data for the corresponding parabolic differential-difference equation was investigated in [10]. Our new approach allows to describe the space of initial data for various classes of parabolic functional differential equations without any additional restrictions on $\mathcal{D}(\mathcal{A})$ and the domain Q.

A brief summary of the main results was given in [11].

2 Statement of problem

Let V and H be separable Hilbert spaces. Suppose V is dense in H and the embedding $V \subset H$ is continuous. Denote by V' the adjoint space. Clearly, we have $V \subset H \subset V'$.

We consider a bounded linear operator $\mathrm{A} : V \to V'$.

Definition 2.1 *The operator* A *is called* V*-coercive if for each* $v \in V$

$$\mathrm{Re}\langle \mathrm{A}v, v\rangle \geq c_1 \|v\|_V^2, \tag{2.1}$$

where $c_1 > 0$ *does not depend on* v*.*

Remark 2.1 *By Theorem 9.1 [12, Chapter 1] the operator* A *has a bounded inverse operator.*

We consider the semilinear form $a(u,v)$ in H given by $a(u,v) = \langle \mathrm{A}u, v\rangle$, $\mathcal{D}(a) = V$. By virtue of (2.1), the form a is a closed sectorial form in H. Indeed, let $u \in V$; then we have

$$|\operatorname{Im} a(u,u)| \leq |a(u,u)| \leq c_2 \|u\|_V^2.$$

From this and (2.1) it follows that the numerical range $\Theta(a)$ satisfies

$$\Theta(a) \subset \{\lambda \in \mathbb{C} : -\theta_1 < \arg\lambda < \theta_1\},$$

where $\theta_1 = \operatorname{arctg}(c_2/c_1) < \pi/2$.

We introduce the operator $\mathcal{A} : H \to H$, generated by the form a, (i.e., $(\mathcal{A}u, v)_H = a(u,v)$, $u \in \mathcal{D}(\mathcal{A})$, $v \in V$). Obviously, $\mathcal{A}u = \mathrm{A}u$ whenever $u \in \mathcal{D}(\mathcal{A})$. By virtue of Theorem 2.1 [13, Chapter 6], the unbounded operator $\mathcal{A} : H \to H$ is a closed, dense, sectorial one.

We consider the Hilbert space $\mathcal{D}(\mathcal{A})$ with the inner product $(u,v)_{\mathcal{D}(\mathcal{A})} = (\mathcal{A}u, \mathcal{A}v)_H + (u,v)_H$.

We introduce the Hilbert space $\mathcal{W}(\mathcal{A}) = \{w \in L_2(0,T;\mathcal{D}(\mathcal{A})) : w' \in L_2(0,T;H)\}$ with the inner product

$$(u,v)_{\mathcal{W}(\mathcal{A})} = \int_0^T (u',v')_H \, dt + \int_0^T (\mathcal{A}u,\mathcal{A}v)_H dt + \int_0^T (u,v)_H dt.$$

Here we consider the derivatives in the sense of distributions in $L_2(0,T;H)$.

Consider the differential equation in the Hilbert space H

$$u'(t) + \mathcal{A}u(t) = f(t) \qquad (t \in (0,T)) \tag{2.2}$$

with the initial condition

$$u|_{t=0} = \varphi, \tag{2.3}$$

where $0 < T < \infty$, $f \in L_2(0,T;H)$, and $\varphi \in H$.

Definition 2.2 *A function $u \in \mathcal{W}(\mathcal{A})$ satisfying (2.2), (2.3) is called a strong solution of problem (2.2), (2.3).*

3 Strong solvability

Lemma 3.1 *Let* A *be a coercive operator. Then the operator $-\mathcal{A}$ is a generator of an analytic semigroup $\{T_t\}_{t\geq 0}$ in H.*

Proof: Since the operator $\mathcal{A}$ is a closed, dense, and sectorial one, by virtue of Theorem 1.24 [13, Chapter 9], the operator $-\mathcal{A}$ is a generator of an analytic semigroup T_t in H. ■

We introduce the set $\Phi = \{\varphi \in H$: problem (2.2), (2.3) has a strong solution$\}$.

Definition 3.1 *The set Φ is called a space of initial data for problem (2.2), (2.3).*

We have $\mathcal{D}(\mathcal{A}) \subset \Phi \subset H$. The space of inital data for problem (2.2), (2.3) can be described by means of an interpolation space. So let us introduce the conception of interpolation spaces.

Let H_1 be a Hilbert space, dense in H, with the continuous embedding $H_1 \subset H$. Let $t[u,v]$ be a closed, symmetric, and positive form in H, defined by the formula $t[u,v] = (u,v)_{H_1}$, $\mathcal{D}(t) = H_1$. By Theorem 2.23 [13, Chapter 6], there exists a positive operator T such that $\mathcal{D}(T^{1/2}) = \mathcal{D}(t) = H_1$. Put $\Lambda = T^{1/2}$. We intoduce the interpolation spaces $[H_1;H]_\theta$ $(0 \leq \theta \leq 1)$ by formula $[H_1;H]_\theta = \mathcal{D}(\Lambda^{1-\theta})$ $(0 \leq \theta \leq 1)$. The space $[H_1;H]_\theta$ is a Hilbert space with the inner product

$$(u,v)_{[H_1,H]_\theta} = (\Lambda^{1-\theta}u, \Lambda^{1-\theta}v)_H + (u,v)_H.$$

Theorem 3.1 *Suppose the operator* A *is* V*-coercive.*

Then problem (2.2), (2.3) has a unique strong solution if and only if $\varphi \in [\mathcal{D}(\mathcal{A}), H]_{1/2}$. *Moreover, this solution is given by*

$$u(t) = T_t\varphi + \int\limits_0^t T_{t-s} f(s)ds, \tag{3.1}$$

where $\{T_t\}_{t\geq 0}$ *is the analytic semigroup generated by the operator* $-\mathcal{A}$.

Proof: By virtue of Theorem 3.7 [2, Chapter 1], problem (2.2), (2.3) has a unique strong solution if and only if the following inequality holds:

$$\int\limits_0^T \|\mathcal{A}T_t\varphi\|_H^2 dt < \infty. \tag{3.2}$$

At the same time, Theorem 1.14.5 [14, Chapter 1] implies that inequality (3.2) takes place if and only if $\varphi \in [\mathcal{D}(\mathcal{A}), H]_{1/2}$. ■

Theorem 3.1 shows that $\Phi = [\mathcal{D}(\mathcal{A}), H]_{1/2}$. Clearly, one would like to get some constructive description of the space $[\mathcal{D}(\mathcal{A}), H]_{1/2}$. Generally, this is a very difficult problem since the domaiin of $\mathcal{A}$ may have quite a complicated form.

In the next Section we obtain a natural (and rather general) condition on the domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$, under which we have $\Phi = V$.

4 Interpolation of domain of coercive operator

We consider the adjoint operator $\mathrm{A}^* : V \to V'$. We define the unbounded operator $\mathcal{A}' : \mathcal{D}(\mathcal{A}') \subset H \to H$ with the domain $D(\mathcal{A}') = \{u \in V : \mathrm{A}^* u \in H\}$ by the formula $\mathcal{A}'u = \mathrm{A}^*u$. If the operator A is V-coercive, then the operator A^* is V-coercive, too. For all $u \in \mathcal{D}(\mathcal{A})$, $v \in \mathcal{D}(\mathcal{A}')$ we have

$$(\mathcal{A}u, v)_H = \langle \mathrm{A}u, v\rangle = \langle u, \mathrm{A}^* v\rangle = (u, \mathcal{A}'v)_H. \tag{4.1}$$

Hence $\mathcal{A}' \subset \mathcal{A}^*$ and $\mathcal{A} \subset (\mathcal{A}')^*$. Since the operators A and A^* are V-coercive, we have $0 \notin \sigma(\mathcal{A}) \cup \sigma(\mathcal{A}')$[1]. By virtue of Lemma 13, [15, §6, Chapter 14] concerning adjoint operators, we have $\mathcal{A}^* = \mathcal{A}'$.

Theorem 4.1 *Let the operator* A *be* V*-coercive. If the embeddings* $V \subset [\mathcal{D}(\mathcal{A}); H]_{1/2}$ *and* $V \subset [\mathcal{D}(\mathcal{A}'); H]_{1/2}$ *are continuous, then* $V = [\mathcal{D}(\mathcal{A}); H]_{1/2} = [\mathcal{D}(\mathcal{A}^*); H]_{1/2}$.

Proof: For any $u \in H$ the form $(\mathcal{A}w, u)_H$ defines the linear continuous functional $\langle w, f\rangle = (\mathcal{A}w, u)_H$ on $\mathcal{D}(\mathcal{A})$ since

$$\sup_{w\in\mathcal{D}(\mathcal{A})} \frac{|(\mathcal{A}w, u)_H|}{\|w\|_{\mathcal{D}(\mathcal{A})}} \leq \sup_{w\in\mathcal{D}(\mathcal{A})} \frac{\|\mathcal{A}w\|_H \|u\|_H}{(\|\mathcal{A}w\|_H^2 + \|w\|_H^2)^{1/2}} \leq \|u\|_H.$$

[1] $\sigma(\mathcal{A})$ (resp. $\sigma(\mathcal{A}')$) stands for the spectrum of the operator $\mathcal{A}$ (resp. $\mathcal{A}'$).

The functional f has the form $\mathrm{A}_0'u = f$ with $\mathrm{A}_0' : H \to (\mathcal{D}(\mathcal{A}))'$. From the inequality $\|f\|_{(\mathcal{D}(\mathcal{A}))'} \leq \|u\|_H$, it follows that the operator A_0' is bounded.

Now we shall demonstrate that $\mathcal{A}^* \subset \mathrm{A}_0'$. Suppose that $u \in \mathcal{D}(\mathcal{A}^*)$ and $w \in \mathcal{D}(\mathcal{A})$. Denote $\mathrm{A}_0'u = f_1$ and $\mathcal{A}^*u = f_2$. From (4.1) it follows that

$$\langle w, f_1 \rangle = (\mathcal{A}w, u)_H = (w, \mathcal{A}^*u)_H = \langle w, f_2 \rangle.$$

Since $w \in \mathcal{D}(\mathcal{A})$ is arbitrary and $\mathcal{D}(\mathcal{A})$ is dense in H, we have $\mathcal{A}^*u = \mathrm{A}_0'u$. From the inclusion $\mathcal{A}^*u \in H$ and the embedding $H \subset (\mathcal{D}(\mathcal{A}))'$, it follows that $\mathcal{A}^*u = \mathrm{A}_0'u \in H$. Since the operator $\mathcal{A}^* : \mathcal{D}(\mathcal{A}^*) \to H$ is bounded, the operator $\mathrm{A}_0' : \mathcal{D}(\mathcal{A}^*) \to H$ is bounded as well.

By virtue of the interpolation theorem (see Theorem 5.1 [12, Chapter 1]) the operator A_0' being defined as

$$\mathrm{A}_0' : [\mathcal{D}(\mathcal{A}^*); H]_{1/2} \to [H; (\mathcal{D}(\mathcal{A}))']_{1/2}$$

is bounded. By virtue of Theorem 6.2 [12, Chapter 1], we have $[H; (\mathcal{D}(\mathcal{A}))']_{1/2} = ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$. Therefore the operator

$$\mathrm{A}_0' : [\mathcal{D}(\mathcal{A}^*); H]_{1/2} \to ([\mathcal{D}(\mathcal{A}); H]_{1/2})' \tag{4.2}$$

is bounded. Let us demonstrate that $\mathrm{A}_0'u = \mathrm{A}^*u$ for $u \in V$. Suppose $u \in V$ and $w \in \mathcal{D}(\mathcal{A})$. Denote $\mathrm{A}_0'u = f_1$ and $\mathrm{A}^*u = f_2$. We have

$$\langle w, f_1 \rangle = (\mathcal{A}w, u)_H = \langle \mathrm{A}w, u \rangle = \langle w, \mathrm{A}^*u \rangle = \langle w, f_2 \rangle.$$

Since $w \in \mathcal{D}(\mathcal{A})$ is arbitrary, we see that $\mathrm{A}^*u = \mathrm{A}_0'u$. From the inclusion $\mathrm{A}^*u \in V'$ and the embedding $V' \subset (\mathcal{D}(\mathcal{A}))'$, it follows that $\mathrm{A}^*u = \mathrm{A}_0'u \in V'$.

Take arbitrary $f \in V'$. Then we have $u = (\mathrm{A}^*)^{-1} f \in V$ and

$$\|u\|_V \leq c_2 \|f\|_{V'}, \tag{4.3}$$

where $c_2 > 0$ does not depend on f. By the assumption we have $u \in [\mathcal{D}(\mathcal{A}^*); H]_{1/2}$ and

$$\|u\|_{[\mathcal{D}(\mathcal{A}^*); H]_{1/2}} \leq c_3 \|u\|_V, \tag{4.4}$$

where $c_3 > 0$ does not depend on f.

From (4.2) it follows that $\mathrm{A}_0'u = f \in ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$. Inequalities (4.3) and (4.4) imply

$$\|f\|_{([\mathcal{D}(\mathcal{A}); H]_{1/2})'} \leq c_4 \|u\|_{[\mathcal{D}(\mathcal{A}^*); H]_{1/2}} \leq c_4 c_3 \|u\|_V \leq c_4 c_3 c_2 \|f\|_{V'}.$$

Therefore we have continues embedding $V' \subset ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$. By the assumption the space $([\mathcal{D}(\mathcal{A}); H]_{1/2})'$ is continuously embedded in V'. Thus we have $V' = ([\mathcal{D}(\mathcal{A}); H]_{1/2})'$ and, therefore, $V = [\mathcal{D}(\mathcal{A}); H]_{1/2}$.

Similarly, one obtains $V = [\mathcal{D}(\mathcal{A}^*); H]_{1/2}$. ■

Since the operator $-\mathcal{A}$ is a generator of an analytic semigroup, one can define fractional powers of the operator $\mathcal{A}$ (see [3], [16]).

Theorem 4.2 *Suppose all the conditions of Theorem 4.1 hold.*
Then $\mathcal{D}(\mathcal{A}^{1/2}) = \mathcal{D}((\mathcal{A}^*)^{1/2}) = V$.

Proof: We take arbitrary number $\alpha \in (0, 1/2)$. By virtue of Theorem 1.15.2 [14, Chapter 1], the operator $\mathcal{A}^\alpha : [\mathcal{D}(\mathcal{A}), H]_{1/2} \to [\mathcal{D}(\mathcal{A}), H]_{1/2-\alpha}$ is bounded and invertible. From Theorem 4.1 it follows that the operator

$$\mathcal{A}^\alpha : V \to [\mathcal{D}(\mathcal{A}), H]_{1/2-\alpha} \tag{4.5}$$

is bounded.

For any $\alpha \in (0, 1/2)$ we introduce the operator $K_\alpha = \Lambda^{1/2-\alpha}\mathcal{A}^\alpha$, where Λ is a positive operator with the domain $\mathcal{D}(\Lambda) = \mathcal{D}(\mathcal{A})$. From (4.5) it follows that the operators $K_\alpha : V \to H$ are bounded for all $\alpha \in (0, 1/2)$.

For any $\varphi \in \mathcal{D}(\mathcal{A})$ we have

$$\lim_{\alpha\to 1/2} K_\alpha\varphi = \lim_{\alpha\to 1/2} \Lambda^{1/2-\alpha}\mathcal{A}^\alpha\varphi = \lim_{\alpha\to 1/2} \Lambda^{1/2}\Lambda^{-\alpha}\mathcal{A}^{\alpha-1/2}\mathcal{A}^{1/2}\varphi = \mathcal{A}^{1/2}\varphi. \tag{4.6}$$

Here we used Theorem 14.1 [16, Chapter 4] concering the analiticity of the family of the operators $\mathcal{A}^{-t}$, $t \geq 0$. Since $\mathcal{D}(\mathcal{A})$ is dense in H, from (4.6) and the Banach-Steinhaus theorem it follows that the operator

$$\lim_{\alpha\to 1/2} K_\alpha = \mathcal{A}^{1/2} : V \to H$$

is bounded. Therefore $V \subset \mathcal{D}(\mathcal{A}^{1/2})$. Let us show that $\mathcal{D}(\mathcal{A}^{1/2}) \subset V$. From (4.5) it follows that the operators $K_\alpha^{-1} = \mathcal{A}^{-\alpha}\Lambda^{-(1/2-\alpha)} : H \to V$ are bounded. For any $\varphi \in \mathcal{D}(\mathcal{A})$ we have

$$\lim_{\alpha\to 1/2} K_\alpha^{-1}\varphi = \lim_{\alpha\to 1/2} \mathcal{A}^{-\alpha}\Lambda^{-(1/2-\alpha)}\varphi = \mathcal{A}^{-1/2}\varphi.$$

Hence the operator $\mathcal{A}^{-1/2} : H \to V$ is bounded, and we have $\mathcal{D}(\mathcal{A}^{1/2}) \subset V$. So we proved that $\mathcal{D}(\mathcal{A}^{1/2}) = V$. Since the operator $\mathcal{A}^*$ has the same properties as $\mathcal{A}$, analogously to the above we derive that $\mathcal{D}((\mathcal{A}^*)^{1/2}) = V$. ■

Thus Theorem 4.2 is a positive answer for the Kato problem in our case.

5 Spaces of initial data

Let $\mathcal{A}$ be the operator defined in Section 2. Let $C : V \to H$ be a bounded operator. We introduce the unbounded operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset H \to H$ acting by the formula $\mathcal{L}u = \mathcal{A}u + Cu$, $u \in \mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A})$.

We consider the parabolic equation

$$u'(t) + \mathcal{L}u(t) = f(t) \qquad (t \in (0, T)) \tag{5.1}$$

with the initial condition

$$u(0) = \varphi. \tag{5.2}$$

Definition 5.1 *A function $u \in \mathcal{W}(\mathcal{L})$ satisfying (5.1), (5.2) is called a strong solution for problem (5.1), (5.2).*

Theorem 5.1 *Let the operator $\mathcal{A}$ be such that the conditions of Theorem 4.1 hold.*

Then the operator $-\mathcal{L}$ is a generator of an analytic semigroup $\{T_t\}_{t\geq 0}$. Moreover, for any $f \in L_2(0,T;H)$ problem (5.1), (5.2) has a unique strong solution if and only if $\varphi \in V$. This solution is given by

$$u(t) = T_t\varphi + \int_0^t T_{t-s}f(s)ds.$$

Proof: At first let us prove that the operator $-\mathcal{L}$ is a generator of an analytic semigroup. From Lemma 3.1 it follows that the operator $-\mathcal{A}$ is a generator of an analytic semigroup. Since $C : V \to H$ is bounded and $\mathcal{A}$ has a bounded inverse, we have

$$\|Cu\|_H \leq c_1\|u\|_V \leq c_2\|\mathcal{A}u\|_H$$

for all $u \in \mathcal{D}(\mathcal{A})$. Therefore, by virtue of the theorem on perturbation of generators of analytic semigroups, the operator $-\mathcal{L} = -\mathcal{A} - C$ is a generator of an analytic semigroup $\{T_t\}_{t\geq 0}$, $\|T_t\| \leq M$, where M is some positive constant.

Similarly to the proof of Theorem 3.1 we obtain that problem (5.1), (5.2) has a unique strong solution if and only if $\varphi \in [\mathcal{D}(\mathcal{L}); H]_{1/2}$. Since $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{A})$, by virtue of Theorem 4.1 we have $[\mathcal{D}(\mathcal{L}); H]_{1/2} = V$. ∎

Further we consider some concrete realizations of the operator $\mathcal{A}$ (namely, functional differential operators) satisfying the conditions of Theorem 5.1.

To begin with, let us prove one auxiliary result. Let H_1, H_2 be Hilbert spaces. Let H_i be dense in H and the embeddings $H_i \subset H$ continuous ($i = 1, 2$).

Lemma 5.1 *Suppose that the embedding $H_2 \subset H_1$ is continuous.*

Then the embedding $[H_2, H]_{1/2} \subset [H_1; H]_{1/2}$ is also continuous.

Proof: For every $\psi \in H$ and for all $t > 0$ we define the functional

$$K(t, \psi; H_1, H) = \inf_{\substack{\psi_0 + \psi_1 = \psi, \\ \psi_0 \in H_1, \\ \psi_1 \in H}} (\|\psi_0\|_{H_1} + t\|\psi_1\|_H).$$

By virtue of Theorem 15.1 [12, Chapter 1], we have $[H_1; H]_{1/2} = \{\psi \in H : \int_0^\infty t^{-2}\mathrm{K}^2(\mathrm{t}, \psi;\, H_1, H)dt < \infty\}$. Since $H_2 \subset H_1$, we have

$$K(t, \varphi; H_1, H) \leq K(t, \varphi; H_2, H).$$

Hence if $t^{-1}K(t, \varphi; H_2, H) \in L_2(0, \infty)$, then $t^{-1}K(t, \varphi; H_1,\ H) \in L_2(0, \infty)$. Therefore $[H_2, H]_{1/2} \subset [H_1; H]_{1/2}$. ∎

Let $Q \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary ∂Q. Put $Q_T = Q \times (0,T)$.

We denote by $W_2^k(Q)$ the Sobolev space of complex-valued functions with the norm

$$\|u\|_{W_2^k(Q)} = \left\{ \sum_{|\alpha|\le k} \int\limits_Q |\mathcal{D}^\alpha u(x)|^2 dx \right\}^{1/2},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_n^{\alpha_n}$, $\mathcal{D}_i = \frac{\partial}{\partial x_i}$. Denote by $\mathring{W}_2^k(Q)$ the closure of the linear manifold $\dot{C}^\infty(Q)$ (of compactly supported functions, infinitely differentiable in Q) in the space $W_2^k(Q)$. Put $W_2^{-k}(Q) = (\mathring{W}_2^k(Q))'$.

Consider the following case: $H = L_2(Q)$, $V = \mathring{W}_2^1(Q)$, and $V' = W_2^{-1}(Q)$. Introduce the bounded operator $A_B : \mathring{W}_2^1(Q) \to W_2^{-1}(Q)$, defined by the formula $A_B u = -\operatorname{div}(B \operatorname{grad} u)$ with a bounded operator $B : L_2^n(Q) \to L_2^n(Q)$, where $L_2^n(Q) = \prod\limits_{k=1}^n L_2(Q)$.

Denote $\mathring{W}_2^{1,n}(Q) = \prod\limits_{k=1}^n \mathring{W}_2^1(Q)$, $W_2^{1,n}(Q) = \prod\limits_{k=1}^n W_2^1(Q)$.

Assume that the following conditions for the operator B hold.

Condition 5.1 *$Bu \in W_2^1(Q)$ whenever $u \in \mathring{W}_2^1(Q)$, and the operator $B : \mathring{W}_2^{1,n}(Q) \to W_2^{1,n}(Q)$ is bounded.*

Condition 5.2 *The operator A_B is $\mathring{W}_2^1(Q)$-coercive.*

We introduce the unbounded operator $\mathcal{A}_B : \mathcal{D}(\mathcal{A}_B) \subset L_2(Q) \to L_2(Q)$ given by $\mathcal{A}_B u = A_B u$, $u \in \mathcal{D}(\mathcal{A}_B) = \{u \in \mathring{W}_2^1(Q) : A_B u \in L_2(Q)\}$.

Consider the parabolic function differential equation

$$u_t(x,t) + \mathcal{A}_B u(x,t) + Cu(x,t) = f(x,t) \qquad ((x,t) \in Q_T) \tag{5.3}$$

with the boundary condition

$$u|_{\partial Q \times (0,T)} = 0 \tag{5.4}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \in Q). \tag{5.5}$$

Problem (5.3)–(5.5) is a particular case of problem (5.1), (5.2).

At first we remark that from Condition 5.2 for the operator B it follows that Condition 5.2 for the operator B^* holds. Now let us demonstrate that $(\mathcal{A}_B)^* = \mathcal{A}_{B^*}$. Indeed, for all $u, v \in \dot{C}^\infty(Q)$ we have

$$(\mathcal{A}_B u, v)_{L_2(Q)} = -(B\nabla u, \nabla v)_{L_2^n(Q)} = (u, \mathcal{A}_{B^*} v)_{L_2(Q)}. \tag{5.6}$$

Since $\dot{C}^\infty(Q)$ is dense in $\mathring{W}_2^1(Q)$, and $\mathcal{A}_B$ is closed (which follows from its $\mathring{W}_2^1(Q)$-coerciveness) we see that equality (5.6) holds for all $u \in \mathcal{D}(\mathcal{A}_B)$, $v \in$

$\mathcal{D}(\mathcal{A}_{B^*})$. Therefore $\mathcal{A}_{B^*} \subset (\mathcal{A}_B)^*$ and $\mathcal{A}_B \subset (\mathcal{A}_{B^*})^*$. Since the operators $\mathcal{A}_B$ and $\mathcal{A}_{B^*}$ are $\mathring{W}_2^1(Q)$-coercive, it follows that $0 \notin \sigma(\mathcal{A}_B) \cup \sigma(\mathcal{A}_{B^*})$. By virtue of Lemma 13 [15, Section 6, Chapter 14] concerning adjoint operators, we have $(\mathcal{A}_B)^* = \mathcal{A}_{B^*}$.

Since the operator $B : \mathring{W}_2^{1,n}(Q) \to W_2^{1,n}(Q)$ is bounded, then the embeddings $\mathring{W}_2^2(Q) \subset \mathcal{D}(\mathcal{A}_B)$ and $\mathring{W}_2^2(Q) \subset \mathcal{D}(\mathcal{A}_{B^*})$ are continuous.

From Theorem 11.6 [12, Chapter 1], we have $[\mathring{W}_2^2(Q), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$.

Remark 5.1 *Theorem 11.6 [12, Chapter 1] was proved in [12] for the domain Q with smooth boundary. Nevertheless in our case this theorem remains true. The proof is based on Lemma 11.3 [12, Chapter 1]. We should only use the Calderon method of extension of functions from Lipschitz domains instead of the Hestenes method, see Theorem 5 [17, Chapter 6, § 3].*

By virtue of Lemma 5.1 we have $\mathring{W}_2^1(Q) \subset [\mathcal{D}(\mathcal{A}_B); L_2(Q)]_{1/2}$, $\mathring{W}_2^1(Q) \subset [\mathcal{D}(\mathcal{A}_{B^*}); L_2(Q)]_{1/2}$, and the embeddings are continuous. Thus conditions of Theorem 5.1 hold for problem (5.3)–(5.5). Therefore we have the following result.

Lemma 5.2 *Suppose that the operator B satisfies Conditions 5.1 and 5.2. Suppose that the operator B^* satisfies Condition 5.1.*

Then $[\mathcal{D}(\mathcal{A}_B); L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$.

Now we shall show that Conditions 5.1 and 5.2 are quite natural in theory of functional differential equations.

Let us impose some restrictions on the domain Q. Suppose $Q \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial Q = \bigcup_i \overline{M_i}$ $(i = 1, ..., N_0)$, where M_i are $(n-1)$-dimensional manifolds of class C^∞ that are open and connected in the topology of ∂Q. Assume that in a neighborhood of any point $g \in \partial Q \setminus \bigcup_i M_i$ the domain Q is diffeomorphic to an n-dimensional dihedral angle for $n \geq 3$ and to a plane angle for $n = 2$.

We introduce the difference operator $R : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ by the formula

$$Ru(x) = \sum_{h \in M} a_h(x) u(x+h).$$

Here $a_h \in C^\infty(\mathbb{R}^n)$, the set M consists of a finite number of vectors $h \in \mathbb{R}^n$ with integer coordinates.

We introduce the operator $R_Q = P_Q R I_Q : L_2(Q) \to L_2(Q)$, where $I_Q : L_2(Q) \to L_2(\mathbb{R}^n)$ is the operator of extension of functions from $L_2(Q)$ by zero in $\mathbb{R}^n \setminus Q$; $P_Q : L_2(\mathbb{R}^n) \to L_2(Q)$ is the operator of restriction of functions from $\mathbb{R}^n$ to Q.

Let the operator B be equal to the operator $R : L_2^n(Q) \to L_2^n(Q)$ defined by the formula

$$(Ru)_i = \sum_{j=1}^n R_{ijQ}(u)_j, \quad i = 1, 2, \ldots, n.$$

We introduce the unbounded differential-difference operator $\mathcal{A}_R : \mathcal{D}(\mathcal{A}_R) \subset L_2(Q) \to L_2(Q)$ acting in the space of distributions $D'(Q)$ by the formula $\mathcal{A}_R u = A_R u$, $u \in D(\mathcal{A}_R) = \{u \in \mathring{W}_2^1(Q) : \mathcal{A}_R u \in \ L_2(Q)\}$.

To formulate conditions for $\mathring{W}_2^1(Q)$-coerciveness of the operator $\mathcal{A}_R$, we introduce an auxiliary notation, see [9].

Denote by G the additive group generated by M. Let Q_r be the open connected components of the set $Q \setminus \left(\bigcup\limits_{h \in G} (\partial Q + h) \right)$.

Definition 5.2 *The sets Q_r are called subdomains. The set $\mathcal{R}$ of all the subdomains Q_r $(r = 1, 2, \dots)$ is called a decomposition of the domain Q.*

The decomposition $\mathcal{R}$ can be divided into disjoint classes in the following way: $Q_{r_1}, Q_{r_2} \in \mathcal{R}$ belong to the same class if there exists an $h \in G$ such that $Q_{r_2} = Q_{r_1} + h$. We denote the subdomains Q_r by Q_{sl}, where s is a number of class and l is a number of a subdomain in the sth class. Evidently, each class consists of a finite number $N = N(s)$ of subdomains Q_{sl}, and $N(s) \le ([diam Q] + 1)^n$. A set of classes can be countable.

We define the matrices $R_s = R_s(x)$ $(x \in \overline{Q}_{s1})$ of orders $N(s) \times N(s)$ with the elements

$$r_{ij}^s(x) = \begin{cases} a_h(x + h_{si}) & (h = h_{sj} - h_{si} \in M), \\ 0 & (h_{sj} - h_{si} \notin M), \end{cases}$$

where h_{si} is such that $Q_{si} = Q_{s1} + h_{si}$.

Let $x \in \overline{Q}_{s1}$ be an arbitrary point. Consider all the points $x^l \in \overline{Q}$ such that $x^l - x \in M$. Since the domain Q is bounded, the set $\{x^l\}$ consists of a finite number $I = I(s, x)$ of points $(I \ge N(s))$. We shall enumerate the points x^l so that $x^l = x + h_{sl}$ for $l = 1, \dots, N = N(s)$, $x^1 = x$, where h_{sl} satisfies the condition $Q_{sl} = Q_{s1} + h_{sl}$. We introduce the $I \times I$ matrices $A_{ijs}(x)$ with elements $a_{lk}^{ijs}(x)$ given by

$$a_{lk}^{ijs}(x) = \begin{cases} a_{ijh}(x^l), & (h = x^k - x^l \in M), \\ 0, & (x^k - x^l \notin M). \end{cases}$$

We define the $N \times N$ matrix $R_{ijs}(x)$ so that if $N < I$, then $R_{ijs}(x)$ is obtained from $A_{ijs}(x)$ by deleting the last $I - N$ rows and columns. If $N = I$, then $R_{ijs}(x)$ is equal to $A_{ijs}(x)$.

On the one hand, by virtue of Theorem 9.2 [9, Section 9], if for all $s = 1, 2, \dots$, $x \in \overline{Q}_{s1}$, and $0 \ne \xi \in \mathbb{R}^n$ the matrices

$$\sum_{i,j=1}^{n} (A_{ijs}(x) + A_{ijs}^*(x)) \xi_i \xi_j$$

are positive definite, then the operator A_R (which coincides with A_B, where $B = R$) is $\mathring{W}_2^1(Q)$-coercive. On the other hand, if the operator A_R is $\mathring{W}_2^1(Q)$-coercive, then by Theorem 9.1 [9, Section 9], the matrices

$$\sum_{i,j=1}^{n} (R_{ijs}(x) + R_{ijs}^*(x)) \xi_i \xi_j$$

are positive definite for all $s = 1, 2, \dots$, $x \in \overline{Q}_{s1}$, and $0 \neq \xi \in \mathbb{R}^n$.

We consider the parabolic differential-difference equation

$$u_t(x,t) - \mathcal{A}_R u(x,t) = f(x,t) \qquad ((x,t) \in Q_T), \tag{5.7}$$

with the boundary condition

$$u|_{\partial Q \times (0,T)} = 0 \tag{5.8}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \in Q). \tag{5.9}$$

Theorem 5.2 *Suppose that the operator $\mathcal{A}_R$ is $\mathring{W}_2^1(Q)$-coercive.*

Then problem (5.7)–(5.9) has a unique strong solution if and only if $\varphi \in \mathring{W}_2^1(Q)$.

Proof: The operator R satisfies Conditions 5.1 and 5.2. Indeed, by virtue of Lemma 8.13 [9, Section 2], the operators R, R^* maps continuously $\mathring{W}_2^{1,n}(Q)$ into $W_2^{1,n}(Q)$. Under assumption the operator $\mathcal{A}_R$ is $\mathring{W}_2^1(Q)$-coercive. Therefore, by virtue of Lemma 5.2 and Theorem 3.1, problem (5.7)–(5.9) has a unique strong solution if and only if $\varphi \in \mathring{W}_2^1(Q)$ ■

Example 5.1

Let us consider problem (5.7)–(5.9), where $Q = (0, \frac{4}{3}) \times (0, \frac{4}{3})$, $\mathcal{A}_R = -\Delta R_Q$, $R_Q = P_Q R I_Q$, $Ru(x) = u(x) + au(x_1+1, x_2+1) + bu(x_1-1, x_2-1)$, $0 < |a+b| < 2$. Obviously, the decomposition $\mathcal{R}$ of the domain Q consists of the two classes of subdomains: 1) $Q_{11} = (0, \frac{1}{3}) \times (0, \frac{1}{3})$, $Q_{12} = (1, \frac{4}{3}) \times (1, \frac{4}{3})$ and 2) $Q_{21} = Q \setminus (\overline{Q}_{11} \cup \overline{Q}_{12})$. We introduce the set $\mathcal{K} \subset \partial Q$. The set $\mathcal{K}$ consists of 4 points: $g^1 = (\frac{1}{3}, 0)$, $g^2 = (\frac{4}{3}, 1)$, $g^3 = (0, \frac{1}{3})$, $g^4 = (1, \frac{4}{3})$.

The matrices $A_s(x)$ ($x \in \overline{Q}_{s1}; s = 1, 2$) assume the form

$$A_1(x) = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \qquad (x \in \overline{Q}_{11}),$$

$$A_2(x) = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \ (x \in \overline{Q}_{21} \cap \mathcal{K}), \quad A_2(x) = (1) \ (x \in \overline{Q}_{21} \setminus \mathcal{K}).$$

Hence the matrices $(A_s(x) + A_s^*(x))(\xi_1^2 + \xi_2^2)$ ($x \in \overline{Q}_{s1}; s = 1, 2$) are positive definite. Therefore the operator A_R is $\mathring{W}_2^1(Q)$-coercive.

By virtue of Theorem 5.2, problem (3.3)–(3.5) has a unique strong solution if and only if $\varphi \in \mathring{W}_2^1(Q)$.

Unlike elliptic differential equations, smoothness of solutions to the equation

$$\mathcal{A}_R u = f \tag{5.10}$$

can be violated inside the domain Q even for $f \in C^\infty(\overline{Q})$, see [9, Sections 11 and 12]. This means that $\mathcal{D}(\mathcal{A}_R) \neq \mathring{W}_2^1(Q) \cap W_2^2(Q)$. However, by virtue of

Theorem 11.2 [9, Section 11], if $u \in \mathcal{D}(\mathcal{A}_R)$ is a solution of equation (5.10) for $f \in L_2(Q)$, then $u \in W_2^2(Q_{sl} \setminus \mathcal{K}^\varepsilon)$ for every $\varepsilon > 0$ $(s = 1, 2, \dots; l = 1, \dots, N(s))$, where $\mathcal{K}^\varepsilon = \{x \in \mathbb{R}^n : \rho(x, \mathcal{K}) < \varepsilon\}$.

In this example the domain $\mathcal{D}(\mathcal{A}_R)$ cannot be described in the Sobolev space terms. Nevertheless we have got the explicit description of the space of initial data: $[\mathcal{D}(\mathcal{A}_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$.

Now we introduce the operators of contraction and expansion of argument $T_{ij} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ and $T_{ijQ} : L_2(Q) \to L_2(Q)$ by the formulas

$$T_{ij}u(x) = \sum_{l \in N} a_{ijl} u(q^{-l}x), \quad T_{ijQ}v = P_Q T_{ij} I_Q v,$$

where N is a finite set of integer numbers; $a_{ijl} \in \mathbb{C}$; $q > 1$.

We define the operator $T : L_2^n(Q) \to L_2^n(Q)$ by the formula

$$(Tu)_i = \sum_{j=1}^{n} T_{ijQ} u_j \quad (i = 1, 2, \dots, n),$$

where $u = (u_1, \dots, u_n)^T$, $Tu = ((Tu)_1, \dots, (Tu)_n)^T$.

Suppose now that $B = T$. We consider the functional differential operator A_T with contracted and expanded arguments (which coincides with A_B, where $B = T$). Denote $t_{ij}(\lambda) = \sum\limits_{l \in N} a_{ijl} \lambda^l$. By virtue of Theorem 1 [18], if we have

$$\sum_{i,j=1}^{n} t_{ij}(\lambda) \xi_i \xi_j > 0 \quad (\lambda \in \mathbb{C}, |\lambda| = q^{n/2}, 0 \neq \xi \in \mathbb{R}^n), \tag{5.11}$$

then the operator A_T is $\mathring{W}_2^1(Q)$-coercive. If in, addition, we suppose that

$$\overline{Q} \subset qQ,$$

then, by Theorem 2 [18], condition (5.11) is necessary for $\mathring{W}_2^1(Q)$-coercivness of the operator A_T.

We consider the parabolic functional differential equation with contracted and expanded arguments

$$u_t(x, t) - \mathcal{A}_T u(x, t) = f(x, t) \qquad ((x, t) \in Q_T) \tag{5.12}$$

with the boundary condition

$$u|_{\partial Q \times (0,T)} = 0 \tag{5.13}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \in Q). \tag{5.14}$$

Theorem 5.3 *Suppose that condition (5.11) holds.*

Then problem (5.12)–(5.14) has a unique strong solution if and only if $\varphi \in \mathring{W}_2^1(Q)$.

Proof: We claim that the operators T, T^* satisfy Condition 5.1. Indeed, the operators T, T^* continuously map $\mathring{W}_2^{1,n}(Q)$ into $W_2^{1,n}(Q)$. Under the assumptions the operators A_T, A_{T^*} are $\mathring{W}_2^1(Q)$-coercive.

Hence, by virtue of Lemma 5.2 and Theorem 3.1, problem (5.12)–(5.14) has a unique strong solution if and only if $\varphi \in \mathring{W}_2^1(Q)$. ■

Example 5.2

Let $Q = \{|x| < 1\} \subset \mathbb{R}^n$. We consider the parabolic functional differential equation

$$u_t(x,t) - \Delta(u(x,t) + a_1 u(q^{-1}x,t)) = f(x,t) \quad ((x,t) \in Q_T) \tag{5.15}$$

with the boundary condition

$$u|_{\partial Q \times (0,T)} = 0 \tag{5.16}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \in Q) \tag{5.17}$$

where $q > 1$, $a_1 \in \mathbb{R}$.

In this example condition (5.11) means that $|a_1| \le q^{1-n/2}$. By virtue of Theorem 5.3, problem (5.15)–(5.17) has a unique strong solution if and only if $\varphi \in \mathring{W}_2^1(Q)$.

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