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NONLOCAL PARABOLIC PROBLEMS WITH THE SUPPORT OF NONLOCAL TERMS INSIDE A DOMAIN *

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Abstract. In this paper we consider the nonlocal problem with the support of nonlocal terms inside a domain for a parabolic equation. Unique solvability of this problem and smoothness of strong solutions are proved. The proofs are based on the semigroups theory and on the spectral properties elliptic nonlocal problem in Sobolev spaces.

Key Words. Nonlocal problem, parabolic problem, semigroup theory.

AMS(MOS) subject classification. 35K20, 35K90.

1. Introduction. In this paper we study the nonlocal parabolic problem with the support of nonlocal terms inside a domain. Such problems emerge in physics, for example, in nonlinear optics and plasma theory (see [1], [2], [3]). Nonlocal parabolic problems are closely connected with parabolic functional differential equations. Parabolic functional differential equations were studied in [4], [5].

Studying of nonlocal parabolic problems in this paper is based upon the estimation of resolvent for elliptic nonlocal problems obtained by A.L. Skubachevskii, see [6].

2. Statement problem. We consider the parabolic equation

$$u_t(x,t) - \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i}(x,t))_{x_j} + A_1 u(\cdot,t) = f(x,t) \ ((x,t) \in Q_T) \tag{1}$$

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with the nonlocal boundary condition

$$(u(x,t)+B_1u(\cdot,t))_{|\Gamma_T}+B_2u(\cdot,t)=0 \quad ((x,t)\in\Gamma_T) \tag{2}$$

and with the initial condition

$$u_{|t=0}=\varphi(x) \qquad (x\in Q). \tag{3}$$

Here $Q\subset\mathbb{R}^n$ is a bounded domain with boundary $\partial Q\in C^\infty$ (a bounded open interval if $n=1$); $Q_T=Q\times(0,T)$, $\Gamma_T=\partial Q\times(0,T)$, $0<T<\infty$; $a_{ij}\in C^\infty(\mathbb{R}^n)$ $(i,j=1,\dots,n)$ are real-valued functions $\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j>0$, $0\neq\xi\in\mathbb{R}^n$; the operator $A_1:H^1(Q)\to L_2(Q)$ is bounded; the operators B_1,B_2, satisfy conjectures 1, 2.

We assume that the following conditions hold:

Conjecture 1. *$B_1:L_2(Q)\to L_2(Q)$ is a linear bounded operator such that their restriction $B_1:H^2(Q)\to H^2(Q)$ is also a bounded operator, and*

$$\|B_1u\|_{L_2(Q)}\le c_1\|u\|_{L_2(Q_\delta)}, \qquad (u\in L_2(Q))$$

$$\|B_1u\|_{H^2(Q)}\le c_2\|u\|_{H^2(Q_\delta)}, \qquad (u\in H^2(Q))$$

where $Q_\delta=\{x\in Q:\varrho(x,\partial Q)>\delta>0\}$; $c_1,c_2>0$.

Conjecture 2. *$B_2:L_2(Q)\to L_2(\partial Q)$ is a linear bounded operator such that their restriction $B_2:H^{3/2}(Q)\to H^{3/2}(\partial Q)$ is also a bounded operator.*

We introduce the unbounded operator $\mathcal{L}_\gamma:\mathcal{D}(\mathcal{L}_\gamma)\subset L_2(Q)\to L_2(Q)$, by formula $\mathcal{L}_\gamma u=(A_0+A_1)u$, $u\in\mathcal{D}(\mathcal{L}_\gamma)=\{u\in H^2(Q):Bu=0\}$, where $A_0u=-\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i}(x))_{x_j}$, $Bu=(u+B_1u)_{|\partial Q}+B_2u$.

Let $A:\mathcal{D}(A)\subset L_2(Q)\to L_2(Q)$ be a closed operator, and let $\mathcal{D}(A)$ be dense in $L_2(Q)$. We introduce the Hilbert spaces $\mathcal{D}(A)$ and $\mathcal{V}(A)=L_2(0,T;\mathcal{D}(A))$ with the inner products

$$(\varphi,\psi)_{\mathcal{D}(A)}=(A\varphi,A\psi)_{L_2(Q)}+(\varphi,\psi)_{L_2(Q)} \quad (\varphi,\psi\in\mathcal{D}(A)),$$

$$(u,v)_{\mathcal{V}(A)}=\int_0^T(u,v)_{\mathcal{D}(A)}dt.$$

We also define the Hilbert space $\mathcal{W}(A) = \{w \in L_2(0,T;\mathcal{D}(A)) : w_t \in L_2(0,T;L_2(Q))\}$

$$(u,v)_{\mathcal{W}(A)} = (u,v)_{\mathcal{V}(A)} + \int\limits_0^T (u_t,v_t)_{L_2(Q)}dt.$$

Here we consider derivatives in the sense of distributions in Q_T.

Definition 1. *A function $u \in \mathcal{W}(\mathcal{L}_\gamma)$ satisfying (1), (3) is called a strong solution of problem (1)–(3).*

3. Nonlocal boundary conditions with the support of nonlocal trems inside a domain. We will use semigroup theory to study strong solvability.

The spectrum of the operator $\mathcal{L}_\gamma$ was studied in [6].

Theorem 1. *Assume that the Conjectures 1 and 2 hold.*

(a) The spectrum of the operator $\mathcal{L}_\gamma$ is discrete and if $\lambda \notin \sigma(\mathcal{L}_\gamma)$ then the resolvent $R(\lambda;\mathcal{L}_\gamma)$ is a compact operator.

(b) For any $0 < \varepsilon < \pi$ there exists $q > 0$ such that $\sigma(\mathcal{L}_\gamma) \subset \Omega_{\varepsilon,q} = \{\lambda \in \mathbb{C} : |\lambda| < q, |\arg\lambda| < \varepsilon\}$.

(c) If $\lambda \notin \Omega_{\varepsilon,q}$ then

$$\|R(\lambda;\mathcal{L}_\gamma)\| \le \frac{c_3}{|\lambda|}, \tag{4}$$

where $c_3 > 0$.

Proof. (a) and (b) follow from Theorem 21.3 [6, Chapter 5].

We take $f \in L_2(Q)$ and $\lambda \notin \Omega_{\varepsilon,q}$. From Theorem 21.2 [6, Chapter 5] it follows that

$$\Big(\|R(\lambda;\mathcal{L}_\gamma)f\|^2_{H^2(Q)} + |\lambda|^2\|R(\lambda;\mathcal{L}_\gamma)f\|^2_{L_2(Q)}\Big)^{1/2} \le c_4\|f\|_{L_2(Q)}. \tag{5}$$

From here it follows (4). □

Using Theorem 1 and Criterion [7, Chapter 1], we get the next theorem.

Theorem 2. *If the operator $\mathcal{L}_\gamma$ have the dense domain $\mathcal{D}(\mathcal{L}_\gamma)$, then the operator $-\mathcal{L}_\gamma$ is an infinitesimal generator of analytic semigroup $\{T_t\}$ $(t \ge 0)$ in $L_2(Q)$.*

Theorem 3. *Assume that the Conjectures 1 and 2 hold and the domain $\mathcal{D}(\mathcal{L}_\gamma)$ is dense in $L_2(Q)$.*

Then for all $f \in L_2(Q_T)$ and $\varphi \in \mathcal{D}(\mathcal{L}_\gamma)$ problem $(1)-(3)$ has a unique strong solution. Moreover, this solution is given by the formula

$$u(x,t) = T_t\varphi(x) + \int\limits_0^t T_{t-s}f(x,s)ds, \tag{6}$$

where $\{T_t\}$ $(t \geq 0)$ *is the analytic semigroup generated by the operator* $-\mathcal{L}_\gamma$.

Proof. We shall consider problem (1)–(3) as an abstract Cauchy problem for parabolic equation in the space $L_2(Q)$. By virtue of Theorem 3.7 [7, Chapter 1], problem (1)–(3) has a unique strong solution. Moreover, this solution is given by (6). $\square$

Example 1. We consider equation

$$u_t(x,t) - \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i}(x,t))_{x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i}(x,t) + c(x) u(x,t) = f(x,t) \quad ((x,t) \in Q_T) \tag{7}$$

with the nonlocal condition

$$\sum_{s=0}^{S} \gamma_s(x) u(\omega_s(x), t)|_{\partial Q \times (0,T)} = 0 \tag{8}$$

and with the initial condition

$$u_{|t=0} = \varphi(x) \qquad (x \in Q), \tag{9}$$

where $a_{ij}, b_i, c, \gamma_s \in C^\infty(\overline{Q})$ are real-valued functions; ω_s are infinitely differentiable nondegenerate transformations mapping some neighborhood Γ of the boundary ∂Q onto the set $\omega_s(\Gamma)$ so that $\overline{\omega_s(\Gamma)} \subset Q$ if $s > 0$ and $\omega_0(x) = x$, $\gamma_0(x) = 1$. In addition we assume that

$$\bigcap_{s=0}^{S} \overline{\omega_s(\Gamma)} = \emptyset \tag{10}$$

for some neighborhood Γ.

As demonstrated in example 21.1 [6, Chapter 5], nonlocal condition (8) can be formulated as (2) with in $B_2 = 0$.

We define the operator A_1 by formula $A_1 u(x) = \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x) u(x)$. So we have problem (7)–(9) as problem (1)–(3). By virtue of (10) it is easily shown that $\overline{\mathcal{D}(\mathcal{L}_\gamma)} = L_2(Q)$. By virtue of Theorem 3 problem (7)–(9) has a unique strong solution for all $f \in L_2(Q_T)$ and $\varphi \in H^2(Q)$ such that $\sum_{s=0}^{S} \gamma_s(x) \varphi(\omega_s(x))|_{\partial Q} = 0$.

4. Nonlocal boundary conditions in a cylinder. We consider the parabolic equation (1) with the nonlocal boundary value

$$(u(x,t)+B_1^{\mu}u(\cdot,t))|_{\{x_1=s_\mu\}\times G\times(0,T)}+B_2^{\mu}(\cdot,t)u=0$$

(11) $$((x',t)\in G\times(0,T);\mu=1,2)$$

$$u|_{[0,d]\times\partial G\times(0,T)}=0$$

and with initial value (3). Here $Q=(0,d)\times G$, $G\subset\mathbb{R}^{n-1}$ is a bounded domain with boundary $\partial G\in C^\infty$ if $n\geq 3$, $Q_T=Q\times(0,T)$ $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $x'=(x_2,\dots,x_n)\in\mathbb{R}^{n-1}$, $s_1=0$, $s_2=d$; the opertors $B_1^{\mu},\mathrm{B}_2^{\mu}$, $\mu=1,2$ satisfy conjectures 3, 4.

We assume that the following conditions hold:

CONJECTURE 3. *$B_1^{\mu}:L_2(Q)\to L_2(Q)$ is a linear bounded operator such that their restriction $B_1^{\mu}:H^2(Q)\to H^2(Q)$ is also a bounded operator, and*

$$\|B_1^{\mu}u\|_{L_2(Q)}\leq c_1\|u\|_{L_2(\tilde Q_\delta)},\qquad (u\in L_2(Q))$$

$$\|B_1^{\mu}u\|_{H^2(Q)}\leq c_2\|u\|_{H^2(\tilde Q_\delta)},\qquad (u\in H^2(Q))$$

where $\tilde Q_\delta=(\delta;d-\delta)\times G$ $\delta>0$; $c_1,c_2>0$.

CONJECTURE 4. *$B_2^{\mu}:L_2(Q)\to L_2(G)$ is a linear bounded operator such that their restriction $B_2^{\mu}:H_0^{3/2}(Q)\to H_0^{3/2}(G)$ is also a bounded operator.* Here we denote the spaces $H_0^k(Q)$ and $H_0^k(G)$ $k\geq 1$. $H_0^k(Q)$ and $H_0^k(G)$ are the subspaces of functions in $H^k(Q)$ and $H^k(G)$, respectively, whose traces vanish on $[0,d]\times\partial G$ and ∂G, respectively.

We introduce the unbounded operator $\mathcal{L}_g:\mathcal{D}(\mathcal{L}_g)\subset L_2(Q)\to L_2(Q)$, by formula $\mathcal{L}_\gamma u=(A_0+A_1)u$, $u\in\mathcal{D}(\mathcal{L}_g)=\{u\in H_0^2(Q):B^{\mu}u=0,\mu=1,2\}$, where $A_0u=-\sum\limits_{i,j=1}^{n}(a_{ij}(x)u_{x_i}(x))_{x_j}$, $B^{\mu}u=(u+B_1^{\mu}u)_{|x_1=s_\mu}+B_2^{\mu}u$, $\mu=1,2$.

DEFINITION 2. *A function $u\in\mathcal{W}(\mathcal{L}_g)$ satisfying (1), (3) is called a strong solution of problem (1), (11) and (3).*

The spectrum of the operator $\mathcal{L}_g$ was studied also in [6].

THEOREM 4. *Assume that the Conjectures 3 and 4 hold.*

(a) The spectrum of the operator $\mathcal{L}_g$ is discrete and if $\lambda\notin\sigma(\mathcal{L}_g)$ then the resolvent $R(\lambda;\mathcal{L}_\gamma)$ is a compact operator.

(b) For any $0<\varepsilon<\pi$ there exists $q>0$ such that $\sigma(\mathcal{L}_g)\subset\Omega_{\varepsilon,q}$.

(c) If $\lambda\notin\Omega_{\varepsilon,q}$ then

(12) $$\|R(\lambda;\mathcal{L}_g)\|\leq\frac{c_3}{|\lambda|},$$

where $c_3 > 0$.

Proof. (a) and (b) follow from Theorem 22.2 [6, Chapter 5].

We take $f \in L_2(Q)$ and $\lambda \notin \Omega_{\varepsilon,q}$. From Theorem 22.1 [6, Chapter 5] it follows that

$$\Big(\|R(\lambda;\mathcal{L}_g)f\|^2_{H^2(Q)} + |\lambda|^2\|R(\lambda;\mathcal{L}_g)f\|^2_{L_2(Q)}\Big)^{1/2} \le c_4\|f\|_{L_2(Q)}. \tag{13}$$

From here it follows (12). □

Using Theorem 4 and Criterion [7, Chapter 1], we get the next theorem.

THEOREM 5. *If the operator* $\mathcal{L}_g$ *have the dense domain* $\mathcal{D}(\mathcal{L}_g)$, *then the operator* $-\mathcal{L}_g$ *is an infinitesimal generator of analytic semigroup* $\{T_t\}$ $(t \ge 0)$ *in* $L_2(Q)$.

THEOREM 6. *Assume that the Conjectures 3 and 4 hold and the domain* $\mathcal{D}(\mathcal{L}_g)$ *is dense in* $L_2(Q)$.

Then for all $f \in L_2(Q_T)$ *and* $\varphi \in \mathcal{D}(\mathcal{L}_g)$ *problem (1), (11) and (3) has a unique strong solution. Moreover, this solution is given by the formula*

$$u(x,t) = T_t\varphi(x) + \int\limits_0^t T_{t-s}f(x,s)ds, \tag{14}$$

where $\{T_t\}$ $(t \ge 0)$ *is the analytic semigroup generated by the operator* $-\mathcal{L}_g$.

Proof. We shall consider problem (1), (11) and (3) as an abstract Cauchy problem for parabolic equation in the space $L_2(Q)$. By virtue of Theorem 3.7 [7, Chapter 1], problem (1), (11) and (3) has a unique strong solution. Moreover, this solution is given by (14). □

EXAMPLE 2. We consider equation

$$\begin{gathered} u_t(x,t) - \sum_{i,j=1}^n (a_{ij}(x)u_{x_i}(x,t))_{x_j} + \\ \sum_{i=1}^n b_i(x)u_{x_i}(x,t) + c(x)u(x,t) = f(x,t) \quad ((x,t)\in Q_T) \end{gathered} \tag{15}$$

with the nonlocal condition

$$\begin{gathered} u_{|x_1=s_\mu} + \sum_{i=1}^m b_{\mu i}(x')u_{|x_1=d_i} = 0 \quad (\mu = 1,2) \\ u_{|[0,d]\times\partial G\times(0,T)} = 0 \end{gathered} \tag{16}$$

and with the initial condition

$$u_{|t=0} = \varphi(x) \qquad (x \in Q), \tag{17}$$

where $a_{ij}, b_i, c, b_{\mu i} \in C^\infty(\overline{G})$ are real-valued functions.

As demonstrated in example 22.1 [6, Chapter 5], nonlocal conditions (16) can be formulated as (11).

We define the operator A_1 by formula $A_1 u(x) = \sum\limits_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x)u(x)$. So we have problem (15)–(17) as problem (1), (11) and (3). It is easily shown that $\overline{\mathcal{D}(\mathcal{L}_g)} = L_2(Q)$ By virtue of Theorem 6 problem (15)–(17) has a unique strong solution for all $f \in L_2(Q_T)$ and $\varphi \in H^2(Q)$ such that

$$\varphi_{|_{x_1=s_\mu}} + \sum_{i=1}^{m} b_{\mu i}(x')\varphi_{|_{x_1=d_i}} = 0 \quad (\mu = 1, 2)$$
$$\varphi_{|_{[0,d]\times\partial G}} = 0.$$

5. Smoothness of strong solutions. In this section we demonstrate that strong solutions of nonlocal parabolic problem belong to Sobolev spaces.

We denote by $H^{2k,k}(Q_T)$ the Sobolev space of complex-valued functions with the norm

$$\|u\|_{H^{2k,k}(Q_T)} = \left\{ \sum_{|\alpha+2\beta|\le 2k} \int\limits_{Q_T} |\mathcal{D}_x^\alpha \mathcal{D}_t^\beta u(x,t)|^2 dxdt + \int\limits_{Q_T} |u(x,t)|^2 dxdt \right\}^{1/2}.$$

We consider problem (1)–(3).

THEOREM 7. *Suppose all the conditions of Theorem 3 hold. Let u be a strong solution of problem (1)–(3). Then $u \in H^{2,1}(Q_T)$.*

Proof. From equation (1) we have

$$\mathcal{L}_\gamma u(\cdot,t) = F(\cdot,t), \tag{18}$$

where $F(\cdot,t) = f(\cdot,t) - u_t(\cdot,t) \in L_2(Q_T)$ and $u(\cdot,t) \in \mathcal{D}(\mathcal{L}_\gamma)$ for a.e. $t \in (0,T)$. Since $\mathcal{D}(\mathcal{L}_\gamma) \subset H^2(Q)$ and from (18) it follows that

$$u(\cdot,t) \in H^2(Q)$$

and

$$\|u\|_{H^2(Q)} \le c_1 \|F\|_{L_2(Q)} \tag{19}$$

for a.e. $t \in (0,T)$, where $c_1 > 0$ does not depend on t.

From here we have

$$\|u\|^2_{H^{2,0}(Q_T)} \le c_2(\|f\|^2_{L_2(Q_T)} + \|u_t\|^2_{L_2(Q_T)}).$$

Therefore, $u \in H^{2,1}(Q_T)$. ■

Similarly, we have analogous result for strong solutions of problem (1), (11) and (3).

Theorem 8. *Suppose all the conditions of Theorem 6 hold. Let u be a strong solution of problem (1), (11) and (3). Then $u \in H^{2,1}(Q_T)$.*

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