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THE MIXED BOUNDARY VALUE PROBLEM FOR PARABOLIC DIFFERENTIAL-DIFFERENCE EQUATION *

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Abstract. For the mixed boundary value problem for parabolic differential-difference equation, we prove uniqueness and existence of strong solutions and stability of solutions. The proofs are based on the semigroups theory and on the properties of difference operators in Sobolev spaces.

1. Introduction. We consider the first mixed boundary value problem for parabolic differential-difference equation. Under assumptions of minimal smoothness for initial functions, we prove uniqueness and existence of strong solutions. For sufficiently smooth initial functions, the first mixed problem for differential-difference equation was studied in [1]. We note that parabolic functional differential equations have important aplications to nonlinear optics [2]–[4]. On the other hand, parabolic differential-difference equations are closely connected with nonlocal elliptic and parabolic problems arising in plasma theory [5], [6].

2. Properties of Difference Operators. 1. Everywhere henceforth we shall assume that the following condition is satisfied:

CONDITION 2.1. *If* $n = 1$, *let* $Q = (0, d)$. *In the case* $n \geq 2$, *let* $Q \subset \mathbb{R}^n$ *be a bounded domain with a boundary* $\partial Q = \bigcup_i \overline{M_i}$ $(i = 1, ..., N_0)$. *Here* M_i *are the* $(n-1)$*-dimensional manifolds of class* C^∞, *which are closed and connected in the topology of* ∂Q. *Let in the neighbourhood of each point*

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$g \in K = \partial Q \setminus \bigcup\limits_i M_i$ the domain Q be diffeomorphic to an n-dimensional dihedral angle if $n \geq 3$, and to a plane angle if $n = 2$.

We introduce a difference operator $R : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ by the formula

$$Ru(x) = \sum_{h \in M} a_h(x) u(x+h). \tag{2.1}$$

Here $a_h \in C^\infty(\mathbb{R}^n)$, the set M consists of a finite number of vectors $h \in \mathbb{R}^n$ with integer coordinates.

We introduce the operator $R_Q = P_Q R I_Q : L_2(Q) \to L_2(Q)$, where $I_Q : L_2(Q) \to L_2(\mathbb{R}^n)$ is the operator of extension of functions from $L_2(Q)$ by zero in $\mathbb{R}^n \setminus Q$, $P_Q : L_2(\mathbb{R}^n) \to L_2(Q)$ is the operator of restriction of functions from $L_2(\mathbb{R}^n)$ to Q.

Denote by G the additive group generated by M. Let Q_r be the open connected components of the set $Q \setminus \left(\bigcup\limits_{h \in G} (\partial Q + h) \right)$.

Definition 2.1. A set Q_r is called a subdomain. A set $\mathcal{R}$ of all subdomains Q_r $(r = 1, 2, \ldots)$ is called a decomposition of the domain Q.

The decomposition $\mathcal{R}$ can be divided into disjoint classes in the following way: $Q_{r_1}, Q_{r_2} \in \mathcal{R}$ belong to the same class if there exists an $h \in G$ such that $Q_{r_2} = Q_{r_1} + h$. We denote the subdomains Q_r by Q_{sl}, where s is the number of class and l is the number of a subdomain in the sth class. Evidently, each class consists of a finite number $N = N(s)$ of subdomains Q_{sl} and $N(s) \leq ([diam Q] + 1)^n$. A set of classes can be countable.

We define the matrices $R_s = R_s(x)$ $(x \in \overline{Q}_{s1})$ of order $N(s) \times N(s)$ with the elements

$$r_{ij}^s(x) = \begin{cases} a_h(x + h_{si}) & (h = h_{sj} - h_{si} \in M), \\ 0 & (h_{sj} - h_{si} \notin M), \end{cases} \tag{2.2}$$

where h_{si} is such that $Q_{si} = Q_{s1} + h_{si}$.

Since Q is a bounded domain, by virtue of (2.2), the number of different matrices R_s is finite if the coefficients a_h are constants. Let n_1 denote this number, and let R_{s_ν} denote all different matrices R_s $(\nu = 1, \ldots, n_1)$.

Lemma 2.1. *Let the coefficients a_h be constants.*
Then

$$\sigma(R_Q) = \bigcup_{\nu=1}^{n_1} \sigma(\mathrm{R}_{s_\nu}),$$

where $\sigma(R_Q)$ is the spectrum of R_Q.

For a proof, see Lemma 8.7 in [6], Section 8.

2. We introduce the set $\mathcal{K}$ by the formula

$$\mathcal{K} = \bigcup_{h_1,h_2\in G} \{\overline{Q} \cap (\partial Q + h_1) \cap \overline{[(\partial Q + h_2) \setminus (\partial Q + h_1)]}\}, \tag{2.3}$$

Assume that the following condition holds:

Condition 2.2. *$\mu_{n-1}(\mathcal{K} \cap \partial Q) = 0$ and $K \subset \mathcal{K}$.*

Denote by Γ_p the components of the set $\partial Q \setminus \mathcal{K}$, which are open and connected in the topology of ∂Q. By virtue of Condition 2.2, $\Gamma_p \in C^\infty$.

Lemma 2.2. *If $(\Gamma_p + h) \cap \overline{Q} \neq \varnothing$ for some $h \in G$, then either $\Gamma_p + h \subset Q$, or there is $\Gamma_r \subset \partial Q \setminus \mathcal{K}$ such that $\Gamma_p + h = \Gamma_r$.*

For a proof, see Lemma 7.5 in [6], Section 7.

By virtue of Lemma 2.2, we can decompose the set $\{\Gamma_p + h : \Gamma_p + h \subset \overline{Q}, p = 1, 2, \dots; h \in G\}$ into classes in the following manner. The sets $\Gamma_{p_1} + h_1$ and $\Gamma_{p_2} + h_2$ belong to the same class if 1) there exists an $h \in G$ such that $\Gamma_{p_1} + h_1 = \Gamma_{p_2} + h_2 + h$, and 2) in the case $\Gamma_{p_1} + h_1$, $\Gamma_{p_2} + h_2 \subset \partial Q$, the directions of the inner normals to ∂Q at the points $x \in \Gamma_{p_1} + h_1$ and $x - h \in \Gamma_{p_2} + h_2$ coincide. Clearly, a set $\Gamma_p \subset \partial Q$ can be in only one class, and a set $\Gamma_p + h \subset Q$ is in at most two classes. We denote a set $\Gamma_p + h$ by Γ_{rj}, where r is the number of the class and j is the number of an element in a given class ($1 \leq j \leq J = J(r)$). Without loss of generality, we shall suppose that

$$\Gamma_{r1}, \dots, \Gamma_{rJ_0} \subset Q,\ \Gamma_{r,J_0+1}, \dots, \Gamma_{rJ} \subset \partial Q\ (0 \leq J_0 = J_0(r) < J(r)).$$

Lemma 2.3. *For every $r = 1, 2, \dots$, there exists a unique $s = s(r)$ such that $N(s) = J(r)$ and after some renumbering $\Gamma_{rl} \subset \partial Q_{sl}$ $(l = 1, \dots, N(s))$.*

For a proof, see Lemma 7.7 in [6], Section 7.

Example 2.1. We consider an operator $R : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ $(n \geq 2)$ defined by

$$Ru(x) = \sum_{j=-k}^{k} a_j u(x_1 + j, x_2, \dots, x_n). \tag{2.4}$$

Let $Q = (0, d) \times D$, where k is a natural number, $d = k + \theta$, $0 < \theta \leq 1$, $D \in \mathbb{R}^{n-1}$ is a bounded domain (with boundary $\partial D \in C^\infty$ if $n \geq 3$), $a_j \in \mathbb{C}$.

a) Let $\theta = 1$. Then the decomposition $\mathcal{R}$ consists of one class of subdomains: $Q_{sl} = (l-1, l) \times D$ $(l = 1, \dots, k+1)$. Moreover, $h_{1l} = (l-1, 0, \dots, 0)$ $(l = 1, \dots, k+1)$, and

$$\mathrm{R}_1 = \begin{pmatrix} a_0 & a_1 & \dots & a_k \\ a_{-1} & a_0 & \dots & a_{k-1} \\ \dots & \dots & \dots & \dots \\ a_{-k} & a_{-k+1} & \dots & a_0 \end{pmatrix}. \tag{2.5}$$

If $n \geq 3$, then there are three classes of sets Γ_{rl}: 1) $\Gamma_{1,k+1} = \{0\} \times D$, $\Gamma_{1l} = \{l\} \times D$ $(l = 1, \dots, k)$, 2) $\Gamma_{2l} = \{l\} \times D$ $(l = 1, \dots, k+1)$, 3) $\Gamma_{3l} = (l-1, l) \times \partial D$ $(l = 1, \dots, k+1)$. Clearly, $J_0(1) = J_0(2) = k$ and $J_0(3) = 0$. Similarly, if $n = 2$, we have four classes of sets Γ_{rl} (instead of the third class we obtain two classes). For any $n \geq 2$, $\mathcal{K} = \bigcup\limits_{l=0}^{k+1} (\{l\} \times \partial D)$.

b) Let $\theta < 1$. Then the decomposition $\mathcal{R}$ consists of two classes of subdomains: $Q_{1l} = (l-1, l-1+\theta) \times D$ $(l = 1, \dots, k+1)$ and $Q_{2l} = (l-1+\theta, l) \times D$ $(l = 1, \dots, k)$. Moreover, $h_{1l} = (l-1, 0, \dots, 0)$ $(l = 1, \dots, k+1)$ and $h_{1l} = (l-1, 0, \dots, 0)$ $(l = 1, \dots, k)$. The matrix R_1 of order $(k+1) \times (k+1)$ is given by (2.5). The matrix R_2 of order $k \times k$ is obtained from R_1 by deleting the last row and the last column.

If $n \geq 3$, then there are four classes of sets Γ_{rl}: 1) $\Gamma_{1,k+1} = \{0\} \times D$, $\Gamma_{1l} = \{l\} \times G$ $(l = 1, \dots, k)$, 2) $\Gamma_{2l} = \{l-1+\theta\} \times D$ $(l = 1, \dots, k+1)$, 3) $\Gamma_{3l} = (l-1, l-1+\theta) \times D$ $(l = 1, \dots, k+1)$, 4) $\Gamma_{4l} = (l-1+\theta, l) \times D$ $(l = 1, \dots, k)$. Clearly, $J_0(1) = J_0(2) = k$ and $J_0(3) = J_0(4) = 0$. If $n = 2$, then we have six classes of sets Γ_{rl}.

For any $n \geq 2$, $\mathcal{K} = \bigcup\limits_{l=1}^{k+1} [(\{l-1\} \times \partial D) \cup (\{l-1+\theta\} \times \partial D)]$.

Definition 2.2. A function $\varphi \in C(\overline{Q})$ is said to be M-periodic in $\overline{Q}$ if $\varphi(x) = \varphi(x+h)$ for all $x, x+h \in \overline{Q}$ and $h \in M$.

Lemma 2.4. *Let a function $\varphi(x)$ be M-periodic in $\overline{Q}$.*

Then $R_Q(\varphi u) = \varphi R_Q u$ for all $u \in L_2(Q)$.

For a proof, see Lemma 8.10 in [6], Section 8.

3. We denote by $W_2^k(Q)$ the Sobolev space of complex-valued functions with the norm

$$\|u\|_{W_2^k(Q)} = \left\{ \sum_{|\alpha| \leq k} \int\limits_Q |\mathcal{D}^\alpha u(x)|^2 dx \right\}^{1/2},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_n^{\alpha_n}$, $\mathcal{D}_j = -i\frac{\partial}{\partial x_j}$. We denote by $W_2^{k-1/2}(\Gamma)$ the space of traces on Γ with the norm $\|v\|_{W_2^{k-1/2}(\Gamma)} = \inf \|u\|_{W_2^k(Q)}$ $(u \in W_2^k(Q) : u_{|\Gamma} = v)$, where $\Gamma \subset \overline{Q}$ is a smooth $(n-1)$-dimensional manifold. Various ways of introducing equivalent norms in Sobolev spaces of noninteger order can be found in [7], Chapter 1, for example. Denote by $\mathring{W}_2^k(Q)$ the closure of the linear manifold $\dot{C}^\infty(Q)$ (of compactly supported functions infinitely differentiable in Q) in the space $W_2^k(Q)$.

By Condition 2.1, the boundary ∂Q is Lipschitz. Therefore, for the domains we are considering, the theorem on the continuation of functions in Sobolev space (see Theorem 5 in [8], Chapter 6, §3) is true. As is known, by virtue of this theorem it is sufficient to prove many properties of Sobolev spaces for a bounded domain with smooth boundary. We shall therefore use the results without mentioning each time the conditions on the boundary ∂Q. We note that the property that characterizes the space $\mathring{W}_2^k(Q)$ is not a direct consequence of the theorem on continuation of functions. To complete the picture, we give the proof of this property.

Lemma 2.5. *$\mathring{W}_2^k(Q) = W_{2,\mathcal{D}}^k(Q)$, where $W_{2,\mathcal{D}}^k(Q) = \{u \in W_2^k(Q) : \mathcal{D}_\nu^{\mu-1} u|_{\partial Q \setminus K} = 0,\ \mu = 1, \dots, k\}$, $\mathcal{D}_\nu = -i\partial/\partial\nu$, and ν is the unit external normal to ∂Q at the point $x \in \partial Q \setminus K$.*

Proof. The inclusion $\mathring{W}_2^k(Q) \subset W_{2,\mathcal{D}}^k(Q)$ is obvious. Let us prove the reverse inclusion.

We introduce the space $H_0^k(Q)$ as a completion of the set $C_0^\infty(\overline{Q} \setminus K)$ with respect to the norm

$$\|u\|_{H_0^k(Q)} = \left\{ \sum_{|\alpha| \le k} \int_Q \rho^{2(|\alpha|-k)} |\mathcal{D}^\alpha u|^2 dx \right\}^{1/2},$$

where $C_0^\infty(\overline{Q} \setminus K)$ is the set of infinitely differentiable functions in $\overline{Q}$ with compact support belonging to $\overline{Q} \setminus K$; $\rho \in C^\infty(\mathbb{R}^n \setminus K)$ is a real-valued function which in some ε-neighbourhood of the set K coincides with distance to the set K, and $\rho(x) \ge c > 0$ outside this ε-neighbourhood.

We define the function $\eta_\delta \in C^\infty(\mathbb{R}^n)$ as follows: $\eta_\delta(x) = 0$ $(x \in K^\delta = \{x \in \mathbb{R}^n : \rho(x, K) < \delta\})$, $\eta_\delta(x) = 1$ $(x \notin K^{2\delta})$, $0 \le \eta_\delta(x) \le 1$ and $|\mathcal{D}^\beta \eta_\delta(x)| \le k_0 \delta^{-|\beta|}$ $(x \in \mathbb{R}^n; |\beta| \le k)$, where $0 < 2\delta < \varepsilon$. Then for each function $w \in H_0^k(Q)$ we have

$\|w - \eta_\delta w\|_{H_0^k(Q)}^2 \le k_1 \sum_{|\gamma| \le k, |\beta| \le k - |\gamma|} \int_{K^{2\delta} \bigcap Q} \rho^{2(|\beta|+|\gamma|-k)} |\mathcal{D}^\gamma w|^2 |\mathcal{D}^\beta (1 - \eta_\delta)|^2 dx \le$

$$k_2 \sum_{|\gamma|\le k} \int_{K^{2\delta}\bigcap Q} \rho^{2(|\gamma|-k)} |\mathcal{D}^\gamma w|^2 dx.$$

Consequently, $\|w - \eta_\delta w\|_{H_0^k(Q)} \to 0$ as $\delta \to 0$. Obviously, the space $H_0^k(Q)$ is continuously embedded in $W_2^k(Q)$. On the other hand, by the embedding theorems for weighted spaces (see [9], §4 and [10], §10) the space $W_\mathcal{D}^k(Q)$ is continuously embedded in $H_0^k(Q)$. Thus, for any function $u \in W_\mathcal{D}^k(Q)$ we have

$$\|u - \eta_\delta u\|_{W_2^k(Q)} \to 0 \quad \text{as} \quad \delta \to 0. \tag{2.6}$$

By construction, $(\eta_\delta u)(x) = 0$ $(x \in K^\delta \bigcap Q)$. We can therefore assume that $\eta_\delta u \in W_\mathcal{D}^k(\Omega)$, $\operatorname{supp}(\eta_\delta u) \subset \overline{\Omega}$, where $\Omega \subset Q$, $\partial\Omega \in C^\infty$. By Theorem 11.5 from [7], Chapter 1, §11, on the characteristic property of the space $\mathring{W}_2^k(\Omega)$ for domains is a smooth boundary, for any $\delta > 0$ there is a sequence of functions $u_s \in \dot{C}^\infty(Q)$, $\operatorname{supp} u_s \subset \Omega$ such that

$$\|\eta_\delta u - u_s\|_{W_2^k(Q)} \to 0 \quad \text{as} \quad s \to 0. \tag{2.7}$$

By (2.6) and (2.7), $u \in \mathring{W}_2^k(Q)$. □

Lemma 2.6. *Let the coefficients a_h of difference operator R be constants.*

Then for all $u \in \mathring{W}_2^k(Q)$

$$\mathcal{D}^\alpha R_Q u = R_Q \mathcal{D}^\alpha u \quad (|\alpha| \le k).$$

For a proof, see Lemma 8.14 in [6], Section 8.

We assume that the following condition is fulfilled:

Condition 2.3. *For every subdomain Q_{sl} $(s = 1, 2, ...;\ l = 1, ..., N(s))$ and for each $\varepsilon > 0$, there exists an open set $G_{sl} \subset Q_{sl}$ with boundary $\partial G_{sl} \in C^1$ such that $\mu_n(Q_{sl} \setminus G_{sl}) < \varepsilon$ and $\mu_{n-1}(\partial G_{sl} \Delta \partial Q_{sl}) < \varepsilon$.*

3. Parabolic Differential-Difference Equations. 1. We introduce the unbounded differential-difference operator $A_R : \mathcal{D}(A_R) \subset L_2(Q) \to L_2(Q)$ acting in the space of distributions $\mathcal{D}'(Q)$ by the formulas

$$A_R u = -\sum_{i,j=1}^n (R_{ijQ} u_{x_j})_{x_i} + \sum_{i=1}^n R_{iQ} u_{x_i} + R_{0Q} u \quad (u \in \mathcal{D}(A_R)),$$

$$\mathcal{D}(A_R) = \{u \in \mathring{W}_2^1(Q) : A_R u \in L_2(Q)\}. \tag{3.1}$$

Here $R_{ijQ} = P_Q R_{ij} I_Q$, $R_{iQ} = P_Q R_i I_Q$,

$$R_{ij}u(x) = \sum_{h\in M} a_{ijh}(x)u(x+h) \quad (i,j = 1,\dots,n),$$

$$R_i u(x) = \sum_{h\in M} a_{ih}(x)u(x+h) \quad (i = 0,1,\dots,n),$$

$M \subset \mathbb{R}^n$ is a finite number of vectors with integer coordinates, a_{ijh}, $a_{ih} \in C^\infty(\mathbb{R}^n)$.

Definition 3.1. The operator A_R is said to be strongly elliptic if there exist constants $c_1 = c_1(A_R) > 0$ and $c_2 = c_2(A_R) \geq 0$ such that for all $u \in \dot{C}^\infty(Q)$

$$\operatorname{Re}(A_R u, u)_{L_2(Q)} \geq c_1 \|u\|^2_{W_2^1(Q)} - c_2 \|u\|^2_{L_2(Q)}. \tag{3.2}$$

In order to formulate necessary and sufficient conditions of strong ellipticity in an algebraic form, we introduce some notation. Let $x \in \overline{Q}_{s1}$ be an arbitrary point. Consider all points $x^l \in \overline{Q}$ such that $x^l - x \in M$. Since the domain Q is bounded, the set $\{x^l\}$ consists of a finite number of points $I = I(s,x)$ $(I \geq N(s))$. We shall number the points x^l so that $x^l = x + h_{sl}$ for $l = 1,\dots,N = N(s)$, $x^1 = x$, where h_{sl} satisfies the condition $Q_{sl} = Q_{s1} + h_{sl}$. We introduce the $I \times I$ matrices $A_{ijs}(x)$ with elements $a^{ijs}_{lk}(x)$ by the formula

$$a^{ijs}_{lk}(x) = \begin{cases} a_{ijh}(x^l), & (h = x^k - x^l \in M), \\ 0, & (x^k - x^l \notin M) \end{cases} \tag{3.3}$$

We define the $N \times N$ matrix $R_{ijs}(x)$ so that, if $N < I$, then $R_{ijs}(x)$ is obtained from $A_{ijs}(x)$ by deleting the last $I - N$ rows and columns. If $N = I$, then $R_{ijs}(x)$ is equal to $A_{ijs}(x)$.

By virtue of Theorem 9.2 in [6], Section 9, if for all $s = 1,2,\dots$, $x \in \overline{Q}_{s1}$, and $0 \neq \xi \in \mathbb{R}^n$ the matrices

$$\sum_{i,j=1}^n (A_{ijs}(x) + A^*_{ijs}(x))\xi_i\xi_j$$

are positive definite, then the operator A_R is strongly elliptic. On the other hand, if the operator A_R is strongly elliptic, then by Theorem 9.1 in [6], Section 9, the matrices

$$\sum_{i,j=1}^n (R_{ijs}(x) + R^*_{ijs}(x))\xi_i\xi_j$$

are positive definite for all $s = 1, 2, \dots$, $x \in \overline{Q}_{s1}$, and $0 \neq \xi \in \mathbb{R}^n$.

REMARK 3.1. We consider the operator $A_R = AR_Q$, where $R_Q = P_Q R I_Q$, $Ru(x) = \sum\limits_{h\in M} a_h u(x + h)$, $a_h \in \mathbb{R}$, $Av = -\sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} v$, $a_{ij} = a_{ji} \in C^\infty(\mathbb{R}^n)$ are real-valued M-periodic functions in $\overline{Q}$.

Let $\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0$ for all $x \in \overline{Q}_{s1}$, $s = 1, 2, \dots$, and $0 \neq \xi \in \mathbb{R}^n$. Then the operator A_R is strongly elliptic iff the matrices $R_s + R_s^*$ are positive definite for all $s = 1, 2, \dots$ (see Example 9.3 in [6], Section 9). Moreover, in this case $c_2(A_R) = 0$.

2. We consider the differential-difference equation

$$u_t(x,t) + A_R u(x,t) = f(x,t) \qquad ((x,t) \in Q_T) \tag{3.4}$$

with boundary condition

$$u|_{\Gamma_T} = 0 \qquad ((x,t) \in \Gamma_T) \tag{3.5}$$

and initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \in Q), \tag{3.6}$$

where $Q_T = Q \times (0,T)$, $\Gamma_T = \partial Q \times (0,T)$, $0 < T < \infty$, $f \in L_2(Q_T)$, and $\varphi \in L_2(Q)$.

Further we shall suppose that the operator A_R is strongly elliptic. In this case problem (3.4)–(3.6) is called the first mixed problem for parabolic differential-difference equation. Without loss of generality, we can assume that in inequality (3.2) $c_2 = 0$.

Let $A : \mathcal{D}(A) \subset L_2(Q) \to L_2(Q)$ be a closed operator, and let $\mathcal{D}(A)$ be dense in $L_2(Q)$. We introduce the Hilbert spaces $\mathcal{D}(A)$ and $\mathcal{V}(A) = L_2(0,T;\mathcal{D}(A))$ with the inner products

$$(\varphi,\psi)_{\mathcal{D}(A)} = (A\varphi, A\psi)_{L_2(Q)} + (\varphi,\psi)_{L_2(Q)} \quad (\varphi,\psi \in \mathcal{D}(A)),$$

$$(u,v)_{\mathcal{V}(A)} = \int\limits_0^T (u,v)_{\mathcal{D}(A)} dt.$$

We also define the Hilbert space $\mathcal{W}(A) = \{w \in L_2(0,T;\mathcal{D}(A)) : \frac{\partial}{\partial t} w \in L_2(0,T;L_2(Q))\}$

$$(u,v)_{\mathcal{W}(A)} = (u,v)_{\mathcal{V}(A)} + \int\limits_0^T \left(\frac{\partial}{\partial t}u, \frac{\partial}{\partial t}v\right)_{L_2(Q)} dt.$$

Here we consider derivatives in the sense of distributions in Q_T.

DEFINITION 3.2. A function $u \in \mathcal{W}(A_R)$ satisfying (3.4), (3.6) is called a strong solution of problem (3.4)–(3.6).

To prove an existence of strong solutions, we make use of semigroup theory.

DEFINITION 3.3. A strongly continuous semigroup of operators $\{T_t\}$ $(t \geq 0)$ in a Hilbert space H is said to be contractive if $\|T_t\| \leq 1$ $(t \geq 0)$.

DEFINITION 3.4. A strongly continuous semigroup of operators $\{T_t\}$ $(t \geq 0)$ in a Hilbert space H is said to be stable if $\lim\limits_{t\to\infty} \|T_t\varphi\| = 0$ for every $\varphi \in H$.

Denote $\Delta_\omega = \{z \in \mathbb{C} : |\arg z| < \omega\}$, where $0 < \omega$.

DEFINITION 3.5. A family of linear bounded operators $\{T_z\}$ $(z \in \Delta_\omega)$ in H is called analytic semigroup in Δ_ω if 1) the function $z \to T_z$ is analytic in Δ_ω, 2) $T_0 = I$ and $\lim\limits_{z\to 0, z\in\Delta_\omega} T_z x = x$ $(x \in H)$, 3) $T_{z_1+z_2} = T_{z_1}T_{z_2}$ $(z_1, z_2 \in \Delta_\omega)$. A semigroup T_t is said to be analytic if there is an analytic extension T_z of the operator-function T_t into some angle Δ_ω.

THEOREM 3.1. *Let the domain Q satisfy Condition* 2.1. *Assume that the operator A_R is strongly elliptic, and that $c_2(A_R) = 0$.*

Then $-A_R$ is an infinitesimal generator of analytic contractive semigroup $\{T_t\}$ $(t \geq 0)$ in $L_2(Q)$. Moreover, the semigroup $\{T_t\}$ $(t \geq 0)$ in $L_2(Q)$ is stable.

Proof. From Theorems 3.1 and 3.2 in [1] it follows that $-A_R$ is a generator of analytic contractive semigroup $\{T_t\}$ $(t \geq 0)$. On the other hand, Theorem 10.1 from [6], Section 10 implies that the spectrum $\sigma(-A_R)$ is discrete and $\sigma(-A_R) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$. Therefore, by virtue of Stability Theorem in [11], the semigroup $\{T_t\}$ $(t \geq 0)$ is stable. □

Let H_1 and H be Hilbert spaces such that H_1 is dense in H with continuous injection $H_1 \subset H$. For every $\psi \in H$ and for all $t > 0$, we define the functional

$$K(t, \psi; H_1, H) = \inf(\|\psi_1\|^2_{H_1} + t^2\|\psi_2\|^2_H)^{1/2}$$

$$(\psi : \psi = \psi_1 + \psi_2, \psi_1 \in H_1, \psi_2 \in H).$$

We now introduce the interpolation space

$$[H_1, H]_{1/2} = \{\psi \in H : \int_0^\infty t^{-2} K^2(t, \psi; H_1, H) dt < \infty\}$$

with the norm

$$\|\psi\|_{[H_1,H]_{1/2}} = (\|\psi\|_H^2 + \int_0^\infty t^{-2} K^2(t, \psi; H_1, H) dt)^{1/2}.$$

LEMMA 3.1. *Let the domain Q satisfy Condition* 2.1. *Then* $[\mathring{W}_2^2(Q), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$.

Proof. The proof is similar to the proof of Theorem 11.6 in [7], Chapter 1 and is based on Lemma 11.3. in [7], Chapter 1. We should only use the Calderon method of extension for functions from Lipschitz domain instead of the Hestenes method, see Theorem 5 in [8], Chapter 6, §3. □

LEMMA 3.2. *Let the domain Q satisfy Condition* 2.1. *Then $\mathring{W}_2^2(Q)$ is continuously imbedded into $\mathcal{D}(A_R)$.*

Proof. From Lemma 8.13 in [6], Section 8, it follows that the operator $R_{ijQ}\frac{\partial}{\partial x_j}$ is mapping continuously $\mathring{W}_2^2(Q)$ into $W_2^1(Q)$. Therefore $\mathring{W}_2^2(Q) \subset \mathcal{D}(A_R)$ and the operator A_R is mapping continuously $\mathring{W}_2^2(Q)$ into $L_2(Q)$. □

LEMMA 3.3. *Let the domain Q satisfy Condition* 2.1. *Assume also that the operator A_R is strongly elliptic, and that $c_2(A_R) = 0$.*

Then problem (3.4) − (3.6) *has a unique strong solution iff $\varphi \in [\mathcal{D}(A_R), L_2(Q)]_{1/2}$. Moreover, this solution is given by the formula*

$$u(x, t) = T_t\varphi(x) + \int_0^t T_{t-s} f(x, s) ds, \tag{3.7}$$

where $\{T_t\}$ $(t \geq 0)$ is the analytic semigroup generated by the operator $-A_R$.

Proof. We shall consider problem (3.4)–(3.6) as an abstract Cauchy problem for parabolic equation in the space $L_2(Q)$. By virtue of Theorem 3.7 in [12], Chapter 1, problem (3.4)–(3.6) has a unique strong solution iff the following inequality holds:

$$\int_0^T \|A_R T_t \varphi\|_{L_2(Q)}^2 dt < \infty. \tag{3.8}$$

Moreover, this solution is given by (3.7). By Theorem 3.1, the semigroup is analytic and contractive. Therefore Theorem 1.14.5 in [13], Chapter 1 implies that inequality (3.8) takes place iff $\varphi \in [\mathcal{D}(A_R), L_2(Q)]_{1/2}$. $\square$

THEOREM 3.2. *Let the domain Q satisfy Condition 2.1. We also assume that the operator A_R is strongly elliptic, and that $c_2(A_R) = 0$.*

Then for all $f \in L_2(Q)$ and $\varphi \in \mathring{W}_2^1(Q)$ problem (3.4) – (3.6) has a unique strong solution. Moreover, this solution is given by formula (3.7).

Proof. By virtue of Lemma 3.3, it is sufficient to prove that $\mathring{W}_2^1(Q) \subset [\mathcal{D}(A_R), L_2(Q)]_{1/2}$.

Let $\varphi \in \mathring{W}_2^2(Q)$. From Lemma 3.2 it follows the inequality

$$K(t, \varphi; \mathcal{D}(A_R), L_2(Q)) \le cK(t, \varphi; \mathring{W}_2^2(Q), L_2(Q)), \tag{3.9}$$

where $c > 0$. Therefore if $t^{-1}K(t, \varphi; \mathring{W}_2^2(Q), L_2(Q)) \in L_2(0, \infty)$, then $t^{-1}K(t, \varphi;\, \mathcal{D}(A_R), L_2(Q)) \in L_2(0, \infty)$. Hence $[\mathring{W}_2^2(Q), L_2(Q)]_{1/2} \subset [\mathcal{D}(A_R), L_2(Q)]_{1/2}$. Thus, by virtue of Lemma 3.1, $\varphi \in \mathring{W}_2^1(Q) = [\mathring{W}_2^2(Q), L_2(Q)]_{1/2} \subset [\mathcal{D}(A_R), L_2(Q)]_{1/2}$. $\square$

4. Space of Initial Data. 1. By virtue of Lemma 3.3, it is natural to consider the space $[\mathcal{D}(A_R), L_2(Q)]_{1/2}$ as the space of initial data for problem (3.4)–(3.6). From Theorem 3.2 it follows that $\mathring{W}_2^1(Q) \subset [\mathcal{D}(A_R), L_2(Q)]_{1/2}$. In this Section we give sufficient conditions of coincidence for these spaces.

THEOREM 4.1. *Let assumptions of Theorem 3.1 hold, and let $A_R : \mathcal{D}(A_R) \subset L_2(Q) \to L_2(Q)$ be a selfadjoint operator.*

Then $[\mathcal{D}(A_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$.

Proof. The operator A_R is selfadjoint and positive. Therefore there exists a selfadjoint operator $A_R^{1/2}$. Hence, using inequality (3.2) and boundedness of difference operators in the space $L_2(Q)$, we obtain

$$c_1\|u\|^2_{W_2^1(Q)} \le (A_R u, u)_{L_2(Q)} = (A_R^{1/2}u, A_R^{1/2}u)_{L_2(Q)} =$$

$$\sum_{i,j=1}^{n} (R_{ijQ}u_{x_j}, u_{x_i})_{L_2(Q)} + \sum_{i=1}^{n} (R_{iQ}u_{x_i}, u)_{L_2(Q)} +$$

$$(R_{0Q}u, u)_{L_2(Q)} \le c_3\|u\|^2_{W_2^1(Q)} \quad (u \in \dot{C}^\infty(Q)). \tag{4.1}$$

Since the operator $A_R^{1/2} : \mathcal{D}(A_R^{1/2}) \subset L_2(Q) \to L_2(Q)$ is closed, from (4.1) it follows that $\mathcal{D}(A_R^{1/2}) = \mathring{W}_2^1(Q)$. By virtue of Theorem 1.18.10 in [13], Chapter 1, we have $[\mathcal{D}(A_R), L_2(Q)]_{1/2} = \mathcal{D}(A_R^{1/2}) = \mathring{W}_2^1(Q)$. $\square$

THEOREM 4.2. *Let assumptions of Theorem* 3.1 *hold. Let there exist a selfadjoint strongly elliptic differential-difference operator* B_R *of form* (3.1) *such that* $c_2(B_R) = 0$ *and* $\mathcal{D}(A_R)$ *is continuously imbedded into* $\mathcal{D}(B_R)$.

Then $[\mathcal{D}(A_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$.

Proof. By Theorem 3.2, $\mathring{W}_2^1(Q) \subset [\mathcal{D}(A_R), L_2(Q)]_{1/2}$. We prove the inverse inclusion.

Theorem 4.1 implies that $[\mathcal{D}(B_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$. From conditions of the theorem we obtain

$$K(t, \varphi; \mathcal{D}(B_R), L_2(Q)) \le cK(t, \varphi; \mathcal{D}(A_R), L_2(Q)),$$

where $c > 0$. Therefore $[\mathcal{D}(A_R), L_2(Q)]_{1/2} \subset [\mathcal{D}(B_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$. $\square$

2. Unlike elliptic differential equations, the smoothness of solutions to equation

$$A_R u = f \tag{4.2}$$

can be violated inside domain Q even for $f \in C^\infty(\overline{Q})$, see [6], Sections 11 and 12. This means that $\mathcal{D}(A_R) \ne \mathring{W}_2^1(Q) \cap W_2^2(Q)$. However, by virtue of Theorem 11.2 in [6], Section 11, if $u \in \mathcal{D}(A_R)$ is a solution of equation (4.2) for $f \in L_2(Q)$, then $u \in W_2^2(Q_{sl} \setminus \mathcal{K}^\varepsilon)$ for every $\varepsilon > 0$ $(s = 1, 2, \dots; l = 1, \dots, N(s))$, where $\mathcal{K}^\varepsilon = \{x \in \mathbb{R}^n : \rho(x, \mathcal{K}) < \varepsilon\}$. Under additional assumptions on smoothness near the set $\mathcal{K}$, we prove that $[\mathcal{D}(A_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$. $\square$

THEOREM 4.3. *Let Conditions* 2.1 – 2.3 *hold. Let the numbers* S_0 *and* r_0 *of different classes of subdomains* Q_{sl} *and different classes of surfaces* Γ_{rm} *be finite, and let each subdomain* Q_{sl} $(s = 1, \dots, S_0; l = 1, \dots, N(s))$ *be Lipschitz. Assume also that the operator* A_R *is strongly elliptic,* $c_2(A_R) = 0$, *and that every solution* $u \in \mathcal{D}(A_R)$ *of equation* (4.2) *belongs to* $W_2^2(Q_{sl})$ $(s = 1, \dots, S_0; l = 1, \dots, N(s))$.

Then $[\mathcal{D}(A_R), L_2(Q)]_{1/2} = \mathring{W}_2^1(Q)$.

Proof. 1. First we construct an equivalent norm in the space $\mathcal{D}(A_R)$.

Denote by $W_2^{k,\mathcal{R}}(Q)$ the space of functions u such that $u \in W_2^k(Q_{sl})$ $(s = 1, \dots, S_0; l = 1, \dots, N(s))$ with the norm

$$\|u\|_{W_2^{k,\mathcal{W}}(Q)} = \left\{ \sum_{s,l} \|u\|^2_{W_2^k(Q_{sl})} \right\}^{1/2}. \tag{4.3}$$

Clearly, $W_2^k(Q) \subset W_2^{k,\mathcal{W}}(Q)$ and

$$\|u\|_{W_2^k(Q)} = \|u\|_{W_2^{k,\mathcal{W}}(Q)} \quad (u \in W_2^k(Q)). \tag{4.4}$$

We now describe the space $\mathcal{D}(A_R)$. From condition of the theorem it follows that $\mathcal{D}(A_R) \subset \mathring{W}_2^1(Q) \cap W_2^{k,\mathcal{R}}(Q)$. Let $u \in \mathring{W}_2^1(Q) \cap W_2^{k,\mathcal{R}}(Q)$. Then $R_Q u \in W_2^{k,\mathcal{R}}(Q)$. By virtue of Condition 2.3, integrating by parts over G_{sl} and passing to a limit as $\varepsilon \to 0$, we have

$$\int\limits_{Q_{sl}} \sum_{i,j} R_{ijQ} u_{x_j} \overline{v}_{x_i} dx = -\int\limits_{Q_{sl}} \sum_{i,j} (R_{ijQ} u_{x_j})_{x_i} \overline{v} dx +$$

$$\int\limits_{\partial Q_{sl}\setminus\mathcal{K}} \sum_{i,j} (R_{ijQ} u_{x_j}) \overline{v}|_{\partial Q_{sl}\setminus\mathcal{K}} \cos(\nu, x_i) dx' \tag{4.5}$$

for all $v \in \dot{C}^\infty(Q)$, where ν is the unit external normal vector to ∂Q_{sl} at the point $x' \in \partial Q_{sl} \setminus \mathcal{K}$.

Summing over s, l, from (4.5) we obtain

$$\int\limits_Q \sum_{i,j} R_{ijQ} u_{x_j} \overline{v}_{x_i} dx = \int\limits_Q f \overline{v} dx + \sum_{s,l} \int\limits_{\partial Q_{sl}\setminus\mathcal{K}} \sum_{i,j} (R_{ijQ} u_{x_j}) \overline{v}|_{\partial Q_{sl}\setminus\mathcal{K}} \cos(\nu, x_i) dx'. \tag{4.6}$$

Here $f \in L_2(Q)$ is given by the formula $f(x) = -\sum\limits_{i,j} (R_{ijQ} u_{x_j})_{x_i}(x)$ for $x \in Q_{sl}$. From (4.6) it follows that $u \in \mathcal{D}(A_R)$ if and only if

$$\sum_{s,l} \int\limits_{\partial Q_{sl}\setminus\mathcal{K}} \sum_{i,j} (R_{ijQ} u_{x_j}) \overline{v}|_{\partial Q_{sl}\setminus\mathcal{K}} \cos(\nu, x_i) dx' = 0. \tag{4.7}$$

Each set $\partial Q_{sl} \setminus \mathcal{K}$ consists of finite number of smooth surfaces Γ_{rm}. By virtue of Lemma 2.3, for every fixed r there exists a unique $s = s(r)$ such that $N(s) = J(r)$ and after some renumbering $\Gamma_{rl} \subset \partial Q_{sl}$ $(l = 1, \dots, N(s))$. Clearly, there exist $p = p(r)$ and $m = m(r)$ such that $\Gamma_{r1} \subset \partial Q_{pm}$ and $Q_{pm} \neq Q_{s1}$. We also renumber subdomains of the pth class so that $\Gamma_{rl} \subset \partial Q_{sl} \cap \partial Q_{pl} \cap Q$ $(l = 1, \dots, J_0)$.

Then (4.7) will take the form

$$\sum_{i,j} (R_{ijQ} u_{x_j})_{sl}|_{\Gamma_{rl}} \cos(\nu_{rl}, x_i) =$$

(4.8)
$$\sum_{i,j}(R_{ijQ}u_{x_j})_{pl}|_{\Gamma_{rl}}\cos(\nu_{rl},x_i) \quad (r=1,\dots,r_0 : r\in\{r : J_0(r)>0\}; l=1,\dots,J_0).$$

Here $(\cdot)_{sl}$ is a restriction of function to Q_{sl}, ν_{rl} is a unit normal to Γ_{rl} at a point $x\in\Gamma_{rl}$.

We can rewrite (4.8) as following:

$$\sum_{i,j}\sum_{t=1}^{N(s)} r_{lt}^{ijs}(x)(u_{st})_{x_j}|_{\Gamma_{rl}+h_{st}-h_{sl}}\cos(\nu_{rl},x_i) =$$

(4.9)
$$\sum_{i,j}\sum_{m=1}^{N(p)} r_{lm}^{ijp}(x)(u_{pm})_{x_j}|_{\Gamma_{rl}+h_{pm}-h_{pl}}\cos(\nu_{rl},x_i) \ (r=1,\dots,r_0; l=1,\dots,J_0).$$

Here $r_{lt}^{ijs}(x)=a_{ijh}(x)$ if $h=h_{st}-h_{sl}\in M$, $r_{lt}^{ijs}(x)=0$ if $h=h_{st}-h_{sl}\notin M$ (cf.(2.2)).

Therefore a function $u\in \mathring{W}_2^1(Q)\cap W_2^{2,\mathcal{R}}(Q)$ belongs to $\mathcal{D}(A_R)$ iff equalities (4.9) hold. Conditions (4.9) define a closed linear subspace in $\mathring{W}_2^1(Q)\cap W_2^{2,\mathcal{R}}(Q)$. We denote this subspace by $\mathcal{W}_2^{2,\mathcal{R}}(Q)$. Since $0\notin\sigma(A_R)$ and the operator $A_R:\mathcal{W}_2^{2,\mathcal{R}}(Q)\to L_2(Q)$ is bounded, the Banach inverse operator theorem implies that the norms $\|u\|_{\mathcal{D}(A_R)}$ and $\|u\|_{\mathcal{W}_2^{2,\mathcal{R}}(Q)}$ are equivalent in $\mathcal{D}(A_R)$.

2. Now we prove that $[\mathcal{D}(A_R),L_2(Q)]_{1/2}=\mathring{W}_2^1(Q)$.

Since subdomains Q_{sl} are Lipschitz, from interpolation theorems 7.1, 9.1, and 9.2 in [7], Chapter 1 it follows that $[W_2^2(Q_{sl}),L_2(Q_{sl})]_{1/2}=W_2^1(Q_{sl})$. By assumption of the theorem, the number S_0 is finite. Therefore

$$[W_2^{2,\mathcal{R}}(Q),L_2(Q)]_{1/2}=W_2^{1,\mathcal{R}}(Q). \tag{4.10}$$

On the other hand, from the first part of the proof it follows that the imbedding $\mathcal{D}(A_R)\subset W_2^{2,\mathcal{R}}(Q)$ is continuous. Hence

$$[\mathcal{D}(A_R),L_2(Q)]_{1/2}\subset[W_2^{2,\mathcal{R}}(Q),L_2(Q)]_{1/2}. \tag{4.11}$$

Let $\varphi\in[\mathcal{D}(A_R),L_2(Q)]_{1/2}$. Then, by Theorem 1.6.2 from [13], Chapter 1, for every $m>0$ there exists $\varphi_{1m}\in\mathcal{D}(A_R)$ such that

$$\|\varphi-\varphi_{1m}\|_{[\mathcal{D}(A_R),L_2(Q)]_{1/2}}<1/m. \tag{4.12}$$

From (4.10), (4.11), and (4.12) it follows that

$$\|\varphi-\varphi_{1m}\|_{W_2^{1,\mathcal{R}}(Q)}<c/m, \tag{4.13}$$

where $c>0$ does not depend on m.

Since $\mathcal{D}(A_R)\subset \mathring{W}_2^1(Q)$, by virtue of (4.4), for every $m>0$ there exists $\varphi_{2m}\in \dot{C}^\infty(Q)$ such that

$$\|\varphi_{1m}-\varphi_{2m}\|_{W_2^{1,\mathcal{R}}(Q)}=\|\varphi_{1m}-\varphi_{2m}\|_{W_2^1(Q)}<1/m. \tag{4.14}$$

Hence

$$\|\varphi-\varphi_{2m}\|_{W_2^{1,\mathcal{R}}(Q)}<(c+1)/m.$$

Therefore $\varphi_{2m}\to\varphi$ in the space $W_2^{1,\mathcal{R}}(Q)$ as $m\to\infty$. Thus $\{\varphi_{2m}\}$ is the Cauchy sequence in $W_2^{1,\mathcal{R}}(Q)$. By virtue of (4.4) $\{\varphi_{2m}\}$ is also the Cauchy sequence in $\mathring{W}_2^1(Q)$. From uniqueness of limit it follows that $\varphi\in\mathring{W}_2^1(Q)$. □

Example 4.1. Let $n=1$, and let $Q=(0,d)$, where $d=k+\theta$, $0<\theta\le 1$, k is a natural number. We consider the differential-difference operator given by

$$A_Ru(x)=-(R_Qu)''(x)+R_{0Q}u(x)$$

$$(u\in\mathcal{D}(A_R)=\{u\in\mathring{W}_2^1(0,d): R_Qu\in W_2^2(0,d)\}).$$

Here $R_Q=P_QRI_Q$, $R_{0Q}=P_QR_0I_Q$, $Ru(x)=\sum\limits_{i=-k}^{k}a_iu(x+i)$, $R_0u(x)=\sum\limits_{i=-k}^{k}a_{0i}u(x+i)$, $a_i,a_{0i}\in\mathbb{C}$.

Lemma 2.6 implies that $(R_Qu)''=(R_Qu')'$ for $u\in\mathcal{D}(A_R)$. Hence the operator A_R can be represented in form (3.1).

If $\theta=1$, then the decomposition $\mathcal{R}$ for the interval $(0,d)$ consists of one class of subintervals: $Q_{1l}=(l-1,l)$ $(l=1,\dots,k+1)$. If $\theta<1$, then the decomposition $\mathcal{R}$ consists of two classes of subintervals: $Q_{1l}=(l-1,l-1+\theta)$ $(l=1,\dots,k+1)$ and $Q_{2l}=(l-1+\theta,l)$ $(l=1,\dots,k)$.

We define the matrices R_1 and R_{01} of order $(k+1)\times(k+1)$ with the elements

$$r_{ij}^1=a_{j-i},\quad r_{ij}^{01}=a_{0,j-i}\quad (i,j=1,\dots,k+1), \tag{4.15}$$

respectively (cf. (2.2)). If $\theta < 1$, we also consider the matrices R_2 and R_{02}, which are obtained from R_1 and R_{01}, respectively, by deleting the last column and the last line.

Assume that the matrix $R_1+R_1^*$ is positive, and that the matrix $R_{01}+R_{01}^*$ is nonnegative. Then the matrix $R_2+R_2^*$ is positive definite, and the matrix $R_{02}+R_{02}^*$ is nonnegative. Therefore, by virtue of Remark 3.1 and Lemma 2.1, the operator A_R is strongly elliptic and $c_2(A_R)=0$. On the other hand, by Theorem 3.2 in [6], Section 3 on the smoothness of generalized solutions for differential-difference equations, if $u \in \mathcal{D}(A_R)$, then $u \in W_2^2(Q_{sl})$. Thus, by virtue of Theorem 4.3 and Lemma 3.3, problem (3.4)–(3.6) has a unique strong solution in the rectangle $Q_T=(0,d)\times(0,T)$ iff $\varphi \in \mathring{W}_2^1(0,d)$.

Example 4.2 (see Example 2.1). Let $Q=(0,d)\times D$, where $D \subset \mathbb{R}^{n-1}$ ($n \geq 2$) is a bounded domain (with boundary $\partial D \in C^\infty$ if $n \geq 3$), $d=k+\theta$, $0<\theta\leq 1$, k is a natural number. We consider the differential-difference operator A_R given by

$$A_R u(x) = AR_Q u(x) + R_{0Q}u(x)$$

$$(u \in \mathcal{D}(A_R) = \{u \in \mathring{W}_2^1(Q) : A_R u \in L_2(Q)\}).$$

Here $R_Q = P_Q R I_Q$, $R_{0Q}=P_Q R_0 I_Q$, $Ru(x)=\sum\limits_{i=-k}^{k} a_i u(x_1+i,x_2,\dots,x_n)$, $a_i\in\mathbb{C}$, $R_0u(x)=\sum\limits_{i=-k}^{k} a_{0i}u(x_1+i,x_2,\dots,x_n)$, $a_{0i}\in\mathbb{C}$, $Av(x)=-\sum\limits_{i,j=1}^{n}\frac{\partial}{\partial x_i}a_{ij}(x)\times \frac{\partial}{\partial x_j}v(x)$, $a_{ij}=a_{ji}\in C^\infty(\mathbb{R}^n)$ are real-valued 1-periodic functions.

Lemma 2.6 implies that $AR_Q u = -\sum\limits_{i,j=1}^{n}\frac{\partial}{\partial x_i}R_{ijQ}\frac{\partial}{\partial x_j}u$ ($u\in\mathcal{D}(A_R)$), where $R_{ijQ}v=P_Q R_{ij} I_Q$, $R_{ij}=a_{ij}R$. Hence the operator A_R can be represented in form (3.1).

If $\theta=1$, then the decomposition $\mathcal{R}$ consists of one class of subdomains: $Q_{1l}=(l-1,l)\times D$ ($l=1,\dots,k+1$). If $\theta<1$, then the decomposition $\mathcal{R}$ consists of two classes of subdomains: $Q_{1l}=(l-1,l-1+\theta)\times D$ ($l=1,\dots,k+1$) and $Q_{2l}=(l-1+\theta,l)\times D$ ($l=1,\dots,k$).

We define the matrices R_1 and R_{01} by formula (4.15). We also consider the matrices R_2 and R_{02} similarly to Example 4.1.

Let $\sum\limits_{i,j} a_{ij}(x)\xi_i\xi_j>0$ for all $x\in\overline{Q}_{s1}$ ($s=1,2$ if $\theta<1$ and $s=1$ if $\theta=1$) and $0\neq\xi\in\mathbb{R}^n$. Assume also that the matrix $R_1+R_1^*$ is positive definite, and that the matrix $R_{01}+R_{01}^*$ is nonnegative. Then the matrix $R_2+R_2^*$

is positive definite, and the matrix $R_{02} + R_{02}^*$ is nonnegative. Therefore, by virtue of Remark 3.1 and Lemma 2.1, the operator A_R is strongly elliptic and $c_2(A_R) = 0$. On the other hand, by Theorem 23.2 in [6], Section 23 on the smoothness of generalized solutions for elliptic differential-difference equations in a cylinder, if $u \in \mathcal{D}(A_R)$, then $u \in W_2^2(Q_{sl})$. Thus, by virtue of Theorem 4.3 and Lemma 3.3, problem (3.4)–(3.6) has a unique strong solution in $Q_T = Q \times (0,T)$ iff $\varphi \in \mathring{W}_2^1(Q)$.

We consider a particular but very important case of Example 4.2.

Example 4.3. Let $Q = (0,2) \times (0,1)$. We study the differential-difference operator A_R given by

$$A_R u = -\Delta R_Q u \ (u \in \mathcal{D}(A_R) = \{u \in \mathring{W}_2^1(Q) : A_R u \in L_2(Q)\}).$$

Here $R_Q = P_Q R I_Q$, $Ru(x) = u(x_1, x_2) + a_1 u(x_1+1, x_2) + a_{-1} u(x_1 - 1, x_2)$.

The decomposition $\mathcal{R}$ consists of two subdomains $Q_{11} = (0,1) \times (0,1)$ and $Q_{21} = (1,2) \times (0,1)$. The matrix $R_1 = \begin{pmatrix} 1 & a_1 \\ a_{-1} & 1 \end{pmatrix}$.

Assume that $|a_1 + a_{-1}| < 2$, i.e. the matrix $R_1 + R_1^*$ is positive definite. Then from Example 4.2 it follows that problem (3.4)–(3.6) has a unique strong solution in Q_T iff $\varphi \in \mathring{W}_2^1(Q)$.

Example 4.4. Consider the operator A_R given by

$$A_R u = -(R_{1Q} u_{x_1})_{x_1} - (R_{2Q} u_{x_2})_{x_2}$$

$$(u \in \mathcal{D}(A_R) = \{u \in \mathring{W}_2^1(Q) : A_R u \in L_2(Q)\}).$$

Here $R_{iQ} = P_Q R_i I_Q$ $(i = 1,2)$, $R_1 u(x) = 2u(x_1, x_2) + u(x_1, x_2+1) + u(x_1, x_2 - 1)$, $R_2 u(x) = 2u(x_1, x_2) + u(x_1+1, x_2) + u(x_1 - 1, x_2)$, $Q = (0,2) \times (0,2)$.

The decomposition $\mathcal{R}$ consists of four subdomains $Q_{11} = (0,1) \times (0,1)$, $Q_{12} = (1,2) \times (0,1)$, $Q_{13} = (0,1) \times (1,2)$, $Q_{14} = (1,2) \times (1,2)$. From the definition of the set $\mathcal{K}$ (see (2.3)) we have $\mathcal{K} = \{(i,j) : i,j = 0,1,2\}$. Clearly, the matrices $R_{11} = R_{21} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Hence, by Lemma 2.1, the selfadjoint operators $R_{iQ} : L_2(Q) \to L_2(Q)$ are positive definite. Thus it is easy to see that the operator A_R is strongly elliptic and $c_2(A_R) = 0$. By virtue of Theorem 11.2 in [6], Section 11, if $u \in \mathcal{D}(A_R)$, then $u \in W_2^2(Q_{1l} \setminus \mathcal{K}^\varepsilon)$ $(l = 1, \dots, 4)$ for every $\varepsilon > 0$. Moreover, in this case it was proved that $\mathcal{D}(A_R) = \mathring{W}_2^1(Q) \cap W_2^2(Q)$ (see Example 12.2 in [6], Section 12). Therefore Theorem 4.3 and Lemma 3.3 imply that problem (3.4)–(3.6) has unique strong solution in Q_T iff $\varphi \in \mathring{W}_2^1(Q)$. The same statment follows from Theorem 4.1, since the operator A_R is selfadjoint (see Theorem 10.2 in [6], Section 10).

REMARK 4.1. In [14] equation (4.2) was considered in an arbitrary plane domain. A difference operator could have shifts in different directions. It were stated sufficient conditions for smoothness of solutions near the set $\mathcal{K}$ in subdomains Q_{sl}. However, there are examples when smoothness of solutions of equation (4.2) is violated near the set $\mathcal{K}$ (see Example 11.2 in [6], Section 11). It would be very interesting to describe the space of initial data $[D(A_R), L_2(Q)]_{1/2}$ in this case.

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