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On the Existence of Smooth Solutions to the Dyachenko Equations Governing Free-Surface Unsteady Ideal Fluid Flows

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Let an inviscid incompressible fluid occupy a domain in the plane (x, y) bounded by the free surface

$$-\infty < y \le \eta(x, t),$$
$$-\infty < x < \infty, \quad t > 0.$$

Assuming that the fluid flow is potential, we have $\mathbf{v}(x, y, t) = \nabla\Phi(x, y, t)$, where $\mathbf{v}(x, y, t)$ is a two-dimensional velocity field and $\Phi(x, y, t)$ is a potential. Throughout this paper, the gradient operator and Laplacian are applied with respect to the spatial variables x and y only. The incompressibility condition $\operatorname{div}\mathbf{v} = 0$ implies that the velocity potential obeys the Poisson equation

$$\Delta\Phi(x, y, t) = 0. \tag{1}$$

Equation (1) is supplemented with the boundary and initial conditions

$$(\eta_t + \Phi_x\eta_x - \Phi_y)\big|_{y = \eta(x,t)} = 0, \tag{2}$$

$$\left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + gy\right)\bigg|_{y = \eta(x,t)} = 0, \tag{3}$$

$$\Phi_y\big|_{y = -\infty} = 0, \tag{4}$$

$$\eta\big|_{t = 0} = \eta_0(x), \tag{5}$$

$$\Phi\big|_{t = 0} = \Phi_0(x,y), \tag{6}$$

where g is the acceleration of gravity.

The first results on the existence of analytical solutions to these problems were obtained in [1]. The existence of solutions of finite smoothness was proved in [2, 3].

Numerical methods for these problems were discussed in [4–6]. However, the original equations written in terms of (η, Φ) are not very convenient in the numerical modeling of free surface problems.

In [7, 8] conformal transformations were used to study free surface problems. Systems of integro-differential equations solved for the time derivatives were obtained. Equivalent equations, called the Dyachenko equations, were derived in [8]. These equations were found to be very convenient for numerical solution.

In this paper, we prove the existence of analytical solutions to the Dyachenko equations on a sufficiently small time interval. It is shown also that these solutions on the real line belong to Sobolev spaces, which is important for substantiating numerical methods.

For positive s, let $Q_s = \{w = u + iv \in \mathbb{C}: 0 < u < 2\pi, -\infty < v < s\}$ be an unbounded domain and E_s^0 denote the space of square integrable analytic functions in Q_s that are 2π-periodic with respect to u. The space E_s^0 equipped with the inner product

$$(f,g)_{E_s^0} = \int_{Q_s} f(w)\overline{g(w)}dw$$

is a Hilbert space (see [9, ch. 1, Theorem 1]).

Let $f(w) \in E_s^0$. Then $f(w) = \sum_{k=1}^{\infty} f_k e^{-ikw}$, where the series converges in the E_s^0 norm. Since Q_s is unbounded, the functions equal to a constant do not belong to E_s^0. We introduce the space $E_s = E_s^0 + \mathbb{R}$ equipped with the norm $\|f\|_{E_s} = \left(\sum_{k=0}^{\infty}|f_k|^2\right)$.

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The operator P: $E_s \times E_s \to E_s$ is defined as follows. Let $A = \sum_{k=0}^{\infty} a_k e^{-ikw}$ and $B = \sum_{k=0}^{\infty} b_k e^{-ikw}$. Then

$$P(A, B) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k e^{kv} e^{-iku},$$

where $c_k = \sum_{m=k}^{\infty} a_m \bar{b}_{m-k}$.

Consider the Dyachenko equations

$$\begin{aligned} R_t &= i(UR' - U'R), \quad w \in Q_s, \quad t \in (0, T), \\ V_t &= i(UV' - B'R) + g(R-1), \end{aligned} \tag{7}$$

where $U = P(R, V) + P(V, R)$, $B = P(V, V)$, and the prime denotes the derivative with respect to the complex variable w. Equations (7) are subject to the boundary conditions

$$\begin{aligned} &R, V \text{ are } 2\pi\text{-periodic with respect to } u, \\ &|R| \to 1, \quad |V| \to 0, \quad u \to -\infty, \quad v < 0 \end{aligned} \tag{8}$$

and the initial conditions

$$R(w, 0) = R_0(w), \quad V(w, 0) = V_0(w), \quad w \in Q_s. \tag{9}$$

The initial functions are assumed to belong to E_s.

Definition 1. Functions R and V that are analytic with respect to t and take their values in E_s are called an analytical solution to problem (7)–(9) if these functions satisfy (7)–(9).

Theorem 1. *Let* $R_0, V_0 \in E_{s_1}$, *where* $s_1 > 0$. *Then, for any* $0 < s < s_1$, *there is* $T(s) > 0$ *such that problem* (7)–(9) h*as a unique analytical solution* $R, V \in E_s$ for $t < T(s)$.

The proof of the theorem is based on the Nirenberg–Nishida theorem (see [10, p. 220 in the Russian edition]).

In numerical simulation, it is convenient to deal with solutions on the real line in Sobolev spaces.

The relevant function spaces are defined as follows. Let $\Omega = (0, 2\pi) \times (0, T)$ with $0 < T < \infty$ be a rectangle in the plane (u, t). By $W_2^k(\Omega)$, we denote the Sobolev space of complex-valued functions from $L_2(\Omega)$ with all generalized derivatives up to the kth order from $L_2(\Omega)$, equipped with the norm

$$\|u\|_{W_2^k(\Omega)} = \left\{ \sum_{|\alpha| \le k} \int_{\Omega} |D^{\alpha} u(x)|^2 dx \right\}^{\frac{1}{2}}.$$

Theorem 2. *Let* R, V *be an analytical solution for* $t < T$. *Then, at* $v = 0$ $R, V \in W_2^k(\Omega)$ *for any* $k \ge 1$.

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