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Estimate of the Existence Time for Solutions to the Surface Wave Equation

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We study the equations governing free-surface unsteady ideal fluid flows. Problems describing ideal fluid surface waves have considered by numerous authors (see, for example, [1–7]). Specifically, the well-posedness of problems was examined in the spaces of analytic functions and Sobolev spaces and theorems were derived about the existence of solutions on a sufficiently small time interval. On the other hand, it is obvious from their physical interpretation that these problems cannot have global (in time) solutions for all initial values. Numerical experiments also confirm wave breakdown and singularity formation in the solutions in a finite time. The estimation of the existence time of solutions describing ideal-fluid surface waves is a fundamental problem in the nonlinear dynamics of surface waves in oceanology.

We consider a system of integro-differential equations that is equivalent to the free-surface Euler equations. This system was derived by V.E. Zakharov and A.I. Dyachenko, and it is very convenient for numerical simulation (see [6]). Specifically, direct numerical simulation was used to analyze the origin of freak waves, an interesting (and complex) phenomenon in the ocean. It was shown that freak waves arise due to the nonlinear effects in the equations governing free-surface ideal fluid flows. However, mathematical justification is required for these and other numerical experiments. The well-posedness of the Dyachenko equations was established in [7]. In [8] numerical schemes were presented that were proved to converge to an exact solution when the latter exists on the time interval under consideration. In this paper, methods are described for the constructive estimation of the existence time for solutions to the Dyachenko equations with given initial data.

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Consider a free-surface unsteady two-dimensional ideal fluid flow of infinite depth. Let the fluid occupy an x-periodic domain in the plane (x, y) bounded by a free surface:

$$-\infty < y < \eta(x, t),$$

$$0 < x < 2\pi, \quad t \geq 0.$$

Assuming that the fluid flow is potential, we have the following equation for the velocity potential:

$$\Delta\Phi(x, y, t) = 0. \tag{1}$$

Equation (1) is supplemented by the boundary and initial conditions

$$(\eta_t + \Phi_x\eta_x - \Phi_y)\big|_{y = \eta(x, t)} = 0, \tag{2}$$

$$\left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + gy\right)\bigg|_{y = \eta(x, t)} = 0, \tag{3}$$

$$\Phi_y\big|_{y = -\infty} = 0, \tag{4}$$

$$\eta|_{t = 0} = \eta_0(x), \quad \Phi|_{t = 0} = \Phi_0(x, y). \tag{5}$$

Here, g is the acceleration of gravity.

Problem (1)–(5) is too complicated to solve it directly. Instead of this problem, we consider the Dyachenko equations for the functions $R = 1 + \sum_{k=1}^{\infty} r_k e^{-iku}$ and $V = \sum_{k=1}^{\infty} v_k e^{-iku}$ as derived in [6]:

$$\begin{gathered} R_t(u, t) = i(U(u, t)R_u(u, t) - U_u(u, t)R(u, t)), \\ V_t(u, t) = i(U(u, t)V_u(u, t) - B_u(u, t)R(u, t)) \\ + g(R(u, t) - 1), \\ 0 < u < 2\pi, \quad 0 < t < T, \end{gathered} \tag{6}$$

$$R(0, t) = R(2\pi, t), \ V(0, t) = V(2\pi, t), \ 0 < t < T,$$

$$R(u, 0) = R_0(u), \quad V(u, 0) = V_0(n), \quad 0 < u < 2\pi,$$

where $U = P(V\bar{R} + \bar{V}R)$, $B = P(V\bar{V})$, and $P = \frac{1}{2}(I + i\mathcal{H})$. Here, $\mathcal{H}$ is the Hilbert operator defined as

$$\mathcal{H}[f](u) = \frac{1}{2\pi}\text{v.p.}\int_0^{2\pi}\cot\frac{u'-u}{2}f(u')du'.$$

Given complex-valued functions R and V, we can recover η and Ψ (see [6]).

Define the scale of function spaces

$$E_s = \left\{ f = \sum_{k=0}^{\infty} f_k e^{-iku} : \sum_{k=0}^{\infty} |f_k|^2 e^{2ks} < \infty \right\}, \quad s > 0.$$

This space consists of functions that can be analytically extended to the domain $\{z \in \mathbb{C}: \operatorname{Im} z < s\}$. The solution to problem (6) is a pair of functions $R, V \in C^1([0, T]; E_s)$ that satisfy (6).

Given a function $W \in E_s$, let w_k denote its Fourier coefficients. On elements of E_s, we introduce the function $\nu(W) = \limsup_{k \to \infty} \frac{\ln|w_k|}{k}$, which can take the value $\nu = -\infty$. If there is an index K such that $w_k = 0$ for $k > K$, we set $\nu(W) = -\infty$. On E_s we also define the function

$$\nu_k(W) = \begin{cases} \frac{\ln|w_k|}{k}, & w_k \neq 0 \\ -\infty, & w_k = 0. \end{cases}$$

Theorem 1. *Let* $\lim_{N \to \infty} \| W^N - W \|_{E_s} = 0$ *for* $W \in E_s$. *Assume that* $\nu(W) = \lim_{k \to \infty} \frac{\ln|w_k|}{k} = -s \neq -\infty$.

Then, for any $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *independent of N or k such that*

$$|\nu_k(W^N) - \nu(W)| \le C_\varepsilon \frac{1}{k} + \varepsilon, \quad k \le N$$

except k and N, for which $W = 0$.

Applying Theorem 1, we can estimate the existence time for solutions to problem (6). Let the initial data R_0 and V_0 be in E_{s_1}, $s_1 > 0$. By Theorem 1 in [7], for any $0 < s_0 < s_1$, there is $T_{s_0} > 0$ such that problem (6) has a unique solution $R, V \in E_{s_0}$ on $[0, T_{s_0}]$. Methods for constructing approximate solutions R^N and V^N can be found in [8]. Thus, along with R^N and V^N, we can calculate $\nu(R^N)$ and $\nu(V^N)$. Since C_ε is independent of N or k, the functions $\nu(R)$ and $\nu(V)$ can be estimated so that

$$\max\{|\nu(R)|, |\nu(V)|\} > s_0 + \varepsilon. \qquad (7)$$

The existence theorems guarantee the existence of a solution on the interval $[0; T]$, on which condition (7) holds.

This method for estimating the existence time has been successfully applied at the Shirshov Institute of Oceanology, Russian Academy of Sciences, and the Landau Institute for Theoretical Physics, Russian Academy of Sciences.

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