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SHORT COMMUNICATIONS

Approximation of Evolution Differential Equations in Scales of Hilbert Spaces

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Evolution differential equations in Hilbert spaces arise in numerous problems in different fields of natural sciences. In these problems, it is necessary to approximate infinite-dimensional systems by finite-dimensional equations (or by systems of ordinary differential equations). In the approximation of the evolution equations in scales of Hilbert spaces, high efficiency can be attained by using computers. In the present paper, we consider the problem of estimating the time of existence of solutions and approximations in the chosen scales of spaces. The results obtained allow us to develop constructive methods for obtaining such estimates in specific cases. We note that the problem of estimating the solution existence time in scales of spaces is actual both in the case of linear equations, where global solvability is possible, and in the case of nonlinear systems, where the existence time of the solution is finite.

Let H be a Hilbert space. Throughout the paper, the Hilbert spaces are assumed to be separable and infinite-dimensional. We let e_k denote an orthonormal basis in H. We introduce a continuous generally nonlinear operator $A\colon D \to H$, where D is a Hilbert space densely and continuously embedded in H and satisfying the condition $\{e_k\} \subset D$. We consider the Cauchy problem

$$u'(t) = Au(t), \qquad t \in (0,T) \tag{1}$$

$$u(0) = \varphi, \qquad \varphi \in H. \tag{2}$$

Definition 1. A function $u \in L_2(0,T;D)$, $u' \in L_2(0,T;H)$, satisfying (1) and (2) is called a *solution of problem* (1) (2) on $(0,T)$.

Remark 1. It follows from Remark 2.2 in [1] that a function $u \in L_2(0,T;D)$, $u' \in L_2(0,T;H)$, has a trace $u(0) \in H$.

To construct a scale of Hilbert spaces, we consider a system of functions $\{\gamma_k(s)\}$ defined for $k = 1, 2, \dots$ and $s \geq 0$ and satisfying the following conditions for each fixed k: $\gamma_k(s) > 0$, $\gamma_k(0) = 1$, and $\gamma_k(s)$ are continuous and strictly decreasing in s. We introduce the spaces H_s, $s \geq 0$, as the closure of all the linear spans $\sum_{k=1}^{N} u_k e_k$ with respect to the norm

$$\|u\|_{H_s}^2 = \sum_{k=1}^{\infty} |u_k|^2 \gamma_k^{-2}(s), \qquad \text{where} \quad u_k = (u, e_k)_H.$$

In the space H, we introduce a finite-dimensional projection operator P_N by the formula

$$P_N\left(\sum_{k=1}^{\infty} u_k e_k\right) = \sum_{k=1}^{N} u_k e_k.$$

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We approximate problem (1), (2) by the finite-dimensional problems

$$(u^N)'(t) = P_N A u^N(t), \qquad t \in (0,T) \tag{3}$$

$$u^N(0) = P_N\varphi, \qquad \varphi \in H. \tag{4}$$

Assumption. There exists numbers $s_l > 0$ and $t_l > 0$, $l = 0, 1, \dots$, such that, for all N, problem (3) (4) has a unique solution $u^N(t) \in H_{s_l}$ for $t \in (0, t_N)$, problem (1) (2) has a unique solution u on $(0, t_0)$, and this solution belongs to H_{s_0} for almost all $t \in (0, t_0)$. In this case,

$$\lim_{N\to\infty} \|u^N - u\|_{L_2(0,T;H_{s_0})} = 0$$

for some $T > 0$.

For any element $u \in H_s$, $s > 0$, we introduce the function

$$\nu(u) = \varliminf_{k\to\infty} = \nu_k(u_k),$$

where the function $\nu_k(u_k)$ satisfies the condition

$$\gamma_k(\nu_k(u_k)) = |u_k| \qquad \text{for} \quad 1 \ge |u_k| > 0;$$

we set $\nu(u_k) = 0$ for $u_k > 1$; and we have $\nu(0) = \infty$ for $u_k = 0$.

Theorem 1. *If $u \in H_s$, then $\nu(u) \ge s$.*

Proof. We have the representation $|u_k| = \alpha_k \gamma_k(s)$, where $\alpha_k \in l_2$. Without loss of generality, we assume that $\alpha_k < 1$. Since the function $\nu_k(a)$ does not increase, we have

$$\nu_k(|u_k|) = \nu_k(\alpha_k\gamma_k(s)) \ge \nu_k(\gamma_k(s)) = s. \quad \square$$

Theorem 2. *Suppose that the series*

$$\sum_{k=1}^{\infty} \left(\frac{\gamma_k(s_2)}{\gamma_k(s_1)}\right)^2$$

converges for $0 < s_1 < s_2$; then, for $u \in H_0$ such that $\nu(u) = s$, $0 < s < \infty$,

$$u \in H_{s-\varepsilon} \qquad \textit{for any} \quad \varepsilon \in (0, s).$$

Proof. Let $\varepsilon \in (0, s)$ be fixed. We denote $\delta_k = \nu_k(u_k) - s$. Since

$$\varliminf_{k\to\infty} \nu_k(u_k) = \nu(u) = s,$$

there exists a $K > 0$ such that $\delta_k > -\varepsilon$ for $k \ge K$. In this case, we have $|u_k| = \gamma_k(s + \delta_k)$, $k \ge K$. It follows from the condition on $\gamma_k(s)$ that

$$\|u\|^2_{H_{s-\varepsilon}} = \sum_{k=1}^{K-1} |u_k|^2 \gamma_k^{-2}(s-\varepsilon) + \sum_{k=K}^{\infty} \left(\frac{\gamma_k(s_2)}{\gamma_k(s_1)}\right)^2 < \infty. \quad \square$$

For the practically important case in which $\gamma_k(s) = e^{-ks}$, we obtain an estimate of the function $\nu(u)$ from the values of $\nu_k(u_k^N)$.

Theorem 3. *Let $\gamma_k(s) = e^{-ks}$; suppose that the Assumption is satisfied, the functions u^N are solutions of problem (3) (4), and the function u is a solution of problem (1) (2). For $t \in (0, T)$, where T is the quantity from the Assumption, if $\nu(u(t)) = s$, $0 < s < \infty$, then, for any $0 < \varepsilon < s$, there exists a constant $C_\varepsilon > 0$ independent of N and k and satisfying the inequality*

$$\nu_k(u_k^N(t)) - \nu(u(t)) \le C_\varepsilon \frac{1}{k} + \varepsilon, \qquad k \le N.$$

The proof of Theorem 3 generalizes the proof of Theorem 2.1 [2].

As an example, we consider abstract parabolic equations. Let V be a Hilbert space densely and continuously embedded in H. We consider a continuous linear operator $L\colon V \to V'$, which is V-coercive, i.e., $\operatorname{Re}\langle Lv, v\rangle \ge \|v\|_V^2$ for any $v \in V$. We introduce a closed unbounded operator $L\colon D(L) \subset H \to H$, where

$$D(L) = \{u \in V : Lu \in H\},$$

and introduce the operator in (1) as follows:

$$Au(t) = -Lu(t) + f(t), \qquad \text{where} \quad f \in L_2(0, T; H).$$

In [1] and [3], examples of V-coercive operators are given from the class of functional differential operators which have important applications (see [4]). Using the results of the papers listed above, we can show that the Assumption is satisfied for $\varphi \in V$.

Another important example is given by nonlinear Cauchy–Kowalewski systems. Such systems in Hilbert spaces were studied in numerous works (see, e.g., [5]–[8]). In [2], Theorem 3 was used to obtain constructive estimates for the time of existence of analytic solutions to nonlinear Cauchy–Kowalewski systems. In [9], these results allowed us to solve the important problem of the existence of equations describing the surface waves of an ideal fluid on a finite time interval.

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