ABOUT ANALYTIC SOLVABILITY OF NONSTATIONARY FLOW OF IDEAL FLUID WITH A FREE SURFACE

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1. Introduction

The first results on the existence of analytical solutions to these problems were obtained in [1]. The existence of finitely smooth solutions was proved in [2, 3]. The further inquiry was given in [3, 4].

Numerical methods for these problems were discussed in [5–7]. However, the original equations are not very convenient for the numerical modeling of free surface problems. In [8–10] conformal transformations were used to study free surface problems. Systems of integro-differential equations resolved with respect to the time derivatives were obtained. Equivalent equations, called the Dyachenko equations, were derived in [10]. These equations were found to be very convenient for numerical solution.

In this paper, we prove the existence of analytical solutions to the Dyachenko equations on a sufficiently small interval of time. It is shown also that these solutions on the real axis belong to the Sobolev spaces, which is important for substantiating numerical methods.

A brief summary of the main results was given in [11].

2. Statement of problem

We study a fluid of the infinite depth occupying the area

$$-\infty < y \leqslant \eta(x,t),$$

$$-\infty < x < \infty, \quad t > 0.$$

Suppose that the flow is irrotational; then

$$v(x,y,t) = \nabla \Phi(x,y,t).$$

The condition of incompressibility $\operatorname{div} v = 0$ implies that the velocity potential $\varPhi$ satisfies the Laplace equation

$$\Delta\varPhi(x, y, t) = 0. \quad (1)$$

In absence of an external pressure, the boundary-value conditions imposed on $\varPhi$ and boundary itself are

$$(\eta_t + \varPhi_x\eta_x - \varPhi_y)|_{y=\eta(x,t)} = 0,$$

$$(\varPhi_t + \frac{1}{2}|\nabla\varPhi|^2 + gy)|_{y=\eta(x,t)} = 0,$$

$$\varPhi_y|_{y=-\infty} = 0,$$

$$\eta|_{t=0} = \eta_0(x),$$

$$\varPhi|_{t=0} = \varPhi_0(x, y).$$

This problem (1–2) is very difficult for direct investigation. Following paper [8], we rewrite problem (1–2) in the other form. Let us perform the conformal mapping of the domain

$$-\infty < x < \infty, \quad -\infty < y < \eta(x, t),$$

filled with fluid, to the half-plane

$$-\infty < u < \infty, \quad -\infty < v < 0.$$

After this transformation, the shape of the surface $\eta(x, t)$ can be represented in a parametric form

$$y = y(u, t), \quad x = u + \tilde{x}(u, t),$$

here $\tilde{x}(u, t)$ and $y(u, t)$ are related through the Hilbert transformation

$$y = \mathrm{H}[\tilde{x}], \quad \tilde{x} = -\mathrm{H}[y],$$

$$\mathrm{H}[f](u) = \frac{1}{\pi}\text{v.p.}\int_{-\infty}^{\infty}\frac{f(u')du'}{u'-u}.$$

We introduce the function $\varPsi(x, t) = \varPhi(x, \eta(x, t), t)$. After the conformal mapping we have $\varPhi(x, y, t) \to \varPhi(u, v, t)$, $\varPsi(x, t) \to \varPsi(u, t)$. In was shown in [8], that $y(u, t)$ and $\varPsi(u, t)$ satisfy the following system of equations:

$$y_t = y_u\mathrm{H}\left[\frac{\mathrm{H}[\varPsi_u]}{J}\right] - x_u\frac{\mathrm{H}[\varPsi_u]}{J}, \quad (2)$$

$$\varPsi_t = \frac{\mathrm{H}[\varPsi_u\mathrm{H}[\varPsi_u]]}{J} + \varPsi_u\mathrm{H}\left[\frac{\mathrm{H}[\varPsi_u]}{J}\right] - gy. \quad (3)$$

Here $J = x_u^2 + y_u^2$ is the Jacobian of the mapping.

Equations (2–3) can be written in the complex form. Let functions $z = x + iy$ and $\Phi = \Psi + i\mathrm{H}[\Psi]$ be analytic in the lower half-plane. They satisfy the equations

$$z_t = iUz_u$$

$$\Phi_t = iU\Phi_u - B + ig(z - u),$$

where $U = \mathrm{P}\left[\frac{-\mathrm{H}\Psi_u}{|z_u|^2}\right]$, $B = \mathrm{P}\left[\frac{|\Phi_u|^2}{|z_u|^2}\right]$. Here P is the operator generating a function which is analytic in the lower half-plane, $\mathrm{P} = \frac{1}{2}(\mathrm{I} + i\mathrm{H})$.

The equations (2–3) are resolved for the derivative with respect to t. However this equations are troublesome for a numeric simulation. In particular, a phenomenon of instability appears. Therefore, following paper [10], we introduce two new functions $R(w,t)$ and $V(w,t)$ as follows:

$$R(w,t) = \frac{1}{z_w}, \qquad V(w,t) = i\frac{\Phi_w}{z_w}.$$

Thus, function $R(w,t)$ is analytic in the lower half-plane and has the following boundary-value condition:

$$R(w,t) \to 1, \quad |w| \to \infty, \quad \operatorname{Im} w \leqslant 0.$$

It is obvious that the boundary-value condition for V is

$$V(w,t) \to 0, \quad |w| \to \infty, \quad \operatorname{Im} w \leqslant 0.$$

Then equations for these analytic functions take the following form:

$$R_t = i(UR_w - U_wR) \tag{4}$$

$$V_t = i(UV_w - B_wR) + g(R - 1). \tag{5}$$

Here $U = \mathrm{P}(V\overline{R} + \overline{V}R)$, $B = \mathrm{P}(V\overline{V})$.

Remark 1. For numeric methods we consider this equations in the periodic case.

Equations (4–5) are exact and completely equivalent to the system (1–2). But system (4–5) is much more convenient for analytical and numerical study.

3. Analytic solvability

Let Q be a bounded domain in R^n. We denote by $H^k(Q)$ the Sobolev space of complex-valued functions with the norm

$$\|f\|_{H^k(Q)} = \left\{ \sum_{|\alpha|\leqslant k} \int_Q |\mathrm{d}^\alpha f(x)|^2 dx \right\}^{1/2}.$$

Let $Q_s = \{w = u + iv : 0 < u < 2\pi, |v| < s\}$ be a domain in C, $0 < s < \infty$. We consider the Banach scale E_s. The space E_s consists of restrictions on Q_s of analytical functions in the stripe $\{w \in C : |\operatorname{Im} w| < s\}$. The functions from E_s are 2π-periodic with respect to u and real for $v = 0$. We consider the following norm in the space E_s:

$$\|f\|_{E_s} = \left(\sup_{|v| \leqslant s} \|f\|^2_{H^1(0,2\pi)} \right)^{1/2}.$$

Here $H^1(0, 2\pi)$ is the first-order Sobolev space. We denote by $\|\cdot\|_s$ the norm in E_s.

Lemma 1. *If $f, g \in E_s$, then $fg \in E_s$ and $\|fg\|_s \leqslant c\|f\|_s\|g\|_s$.*

Proof. At first we estimate the norm $\|fg\|_s$:

$$\|fg\|^2_s \leqslant \sup_{|v| \leqslant s} (\|fg\|^2_{L_2(0,2\pi)} + \|f'g\|^2_{L_2(0,2\pi)} + \|fg'\|^2_{L_2(0,2\pi)}).$$

By the Sobolev embedding theorem, we have

$$\sup_{|v| \leqslant s} \|fg\|^2_{L_2(0,2\pi)} \leqslant c_1 \sup_{|v| \leqslant s} \|f\|^2_{L_2(0,2\pi)} \|g\|^2_{C[0,2\pi]} \leqslant c_2 \|f\|^2_s \|g\|^2_s.$$

Similarly, we have:

$$\sup_{|v| \leqslant s} \|f'g\|^2_{L_2(0,2\pi)} \leqslant c_3 \sup_{|v| \leqslant s} \|f'\|^2_{L_2(0,2\pi)} \|g\|^2_{C[0,2\pi]} \leqslant c_4 \|f\|^2_s \|g\|^2_s,$$

$$\sup_{|v| \leqslant s} \|fg'\|^2_{L_2(0,2\pi)} \leqslant c_5 \sup_{|v| \leqslant s} \|f\|^2_{C[0,2\pi]} \|g'\|^2_{L_2(0,2\pi)} \leqslant c_6 \|f\|^2_s \|g\|^2_s.$$

Lemma 2. *Let functions f_1, f_2, g_1, g_2 belong a ball with radius $M > 0$ in E_s. Then*

$$\|f_1 g_1 - f_2 g_2\|_s \leqslant c(M)(\|f_1 - f_2\|_s + \|g_1 - g_2\|_s).$$

Proof. By virtue of lemma 1, we have

$$\|f_1g_1 - f_2g_2\|_s = \|f_1g_1 - f_1g_2 + f_1g_2 - f_2g_2\|_s \leqslant \|f_1(g_1 - g_2)\|_s + \|(f_1 - f_2)g_2\|_s \leqslant$$
$$c(M)(\|f_1 - f_2\|_s + \|g_1 - g_2\|_s).$$

For periodic functions we introduce the Hilbert operator as follows:

$$\mathrm{H}[f] = \frac{1}{\pi} \text{v.p.} \int_0^{2\pi} f(u') \cot(u' - u) du.$$

Lemma 3. *The Hilbert operator is continuous in E_s and $\|\mathrm{H}f\|_s = \|f\|_s$ for all $f \in E_s$.*

Proof. Let $f \in E_s$. Then

$$\|f\|_s^2 = \sup_{|v|\leqslant s} \Big(\sum_{k=-\infty}^{\infty} (1+k^2)^2 e^{2kv} |f_k|^2 \Big),$$

where f_k are Fourier coefficients. Since as $\mathrm{H}[e^{iku}] = i\,\mathrm{sign}(k) e^{iku}$, then

$$\|\mathrm{H}f\|_s^2 = \sup_{|v|\leqslant s} \Big(\sum_{k=-\infty}^{\infty} (1+k^2)^2 e^{2kv} |-i\,\mathrm{sign}(k) f_k|^2 \Big) = \|f\|_s^2.$$

Now, we rewrite equations (4)–(5) in the real form, setting $v = 0$. Let $R = R_1 + iR_2$, $V = V_1 + iV_2$, then we obtain the following equations set

$$\begin{aligned}
\dot{R}_1 &= U_1' R_2 + U_2' R_1 - U_1 R_2' - U_2 R_1', \\
\dot{R}_2 &= U_1 R_1' + U_2 R_2' - U_1' R_1 + U_2' R_2, \\
\dot{V}_1 &= B_1' R_2 + B_2' R_1 - U_1 V_2' - U_2 V_1' + g(R_1 - 1), \\
\dot{V}_2 &= U_1 V_1' + U_2 V_2' - B_1' R_1 + B_2' R_2 + g R_2,
\end{aligned} \tag{6}$$

where $U_1 = R_1 V_1 + R_2 V_2$, $U_2 = \mathrm{H}[R_1 V_1 + R_2 V_2]$, $B_1 = \frac{1}{2}(V_1^2 + V_2^2)$, $B_2 = \frac{1}{2}\mathrm{H}[V_1^2 + V_2^2]$.

Further we rewrite equations (6) in vector form. We denote by E_s^4 the space $\prod_{l=1}^{4} E_s$. We introduce the operator $F : E_s^4 \to E_s^4$ by the right-hand sides of equations (6). Denote $W = [R_1, R_2, V_1, V_2]^T$. We have the equation

$$\dot{W} = F(W), \tag{7}$$

with initial-value condition

$$W(0) = W_0, \tag{8}$$

and boundary-value conditions

$$R_{10} = 1, \quad R_{10} = 0, \quad V_{10} = 0, \quad V_{20} = 0, \tag{9}$$

where $R_{10}, R_{20}, V_{10}, V_{20}$ are Fourier coefficients of the functions R_1, R_2, V_1, V_2 for $k = 0$.

Definition 1. *An analytical on* $[0, T)$ *function* $W(t) = [R_1(t), R_2(t), V_1(t), V_2(t)]^T$ *satisfying (7)–(9) is called an analytical solution of problem (7)–(9).*

Theorem 1. *Let* $W_0 \in E_{s_1}^4$ *satisfy conditions (9). Then for all* $s_2 \in (0, s_1)$ *there is* $T = T(s_2) > 0$ *such that problem (7)–(9) has a unique analytic solution on* $t \in (0, T)$.

Proof. Choose arbitrary s, s', $0 < s' < s < s_1$. We suppose that $W_1, W_2 \in E_s^4$ and $\|W_1\|_{E_s^4} < M$, $\|W_2\|_{E_s^4} < M$.

Now we verify that the operator F satisfies the condition

$$\|F(W_1) - F(W_2)\|_{E_{s'}^4} \leqslant c(M)\frac{\|W_1 - W_2\|_{E_s^4}}{s - s'}. \tag{10}$$

A function $f \in E_s$ can be represented by the Fourier series $f(w) = \sum\limits_{k=-\infty}^{\infty} f_k e^{ikw}$, hence $f'(w) = \sum\limits_{k=-\infty}^{\infty} (ik) f_k e^{ikw}$. We have $f_k = \overline{f}_{-k}$ because $\operatorname{Im} f = 0$ if $v = 0$. From the estimate $k^2 e^{2|k|s} \leqslant \frac{e^2}{(s-s')^2} e^{2|k|s}$ we obtain

$$\|f'\|_{s'} \leqslant c_1 \frac{\|f\|_s}{s - s'}.$$

Hence by Lemmas 2 and 3 we have (10).

Now we consider an auxiliary problem for W_a in E_s^4

$$\dot{W}_a = F(W_a + W_0), \tag{11}$$

$$W_a(0) = 0. \tag{12}$$

By virtue of (10) we can apply the Nirenberg-Nishida theorem (Theorem, [12, p. 220]) to problem (11), (12). So there is a unique analytical solution of problem (11), (12) on $t \in (0, T)$, where $T = T(s_2) > 0$. Hence the function $W(t) = W_a(t) + W_0$ is analytic solution of (7), (8). We need verify that condition (9) holds as well. By the assumption, W_0 satisfies this condition. On the other hand, if $A \in E_s^4$, then for the function $B = F(A) \in E_{s'}^4$ we have

$$B_{10} = 0,\ B_{20} = 0,\ B_{30} = 0,\ B_{40} = 0.$$

Therefore, the function W satisfies condition (9) and this function is analytic solution of problem (7)–(9).

Theorem 1 guarantees that analytic solution exists on sufficiently small interval of time. For methods of estimating the lifetime of analytic solutions see [13].

In numerical simulation, it is convenient to deal with solutions on the real axis in Sobolev spaces. Let $\Omega = (0, 2\pi) \times (0, T)$, $0 < T < \infty$ be the a rectangle in the plane (u, t). The following theorem is corollary of Theorem 1.

Theorem 2. *Let* $[R_1, R_2, V_1, V_2]^T$ *be an analytical solution for* $t \in [0, T]$*. Then on* $v = 0$ *we have* $R_1, R_2, V_1, V_2 \in H^k(\Omega)$ *for all* $k \geqslant 1$.

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